# Math 24: Winter 2021
# Lecture 5

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# Let's Get Started

1. We should be recording.
2. Remember, it is more comfortable for me if you turn on your video so that I feel like I am talking to real people.
3. I put a link to some brief homework solutions on the assignments page. I'll update the link as the term progresses.
4. You should have turned in your second homework by now.
5. Because of the Monday holiday, we will be meeting in our x-hour tomorrow from 12:30 to 1:20.
6. But first, are there any questions from last time?

# Review

## Theorem

*If $S$ is any subset of a vector space $V$, then $\mathrm{Span}(S)$ is subspace of $V$ containing $S$. Furthermore, any subspace of $V$ containing $S$ must also contain $\mathrm{Span}(S)$. Hence $\mathrm{Span}(S)$ is the smallest subspace of $V$ containing $S$.*

## Definition

A subset $S \subset V$ spans $V$ or *generates* $V$ if $V = \mathrm{Span}(S)$. In this case, we say that vectors of $S$ span or generate $V$.

## Proposition

*Let $v_1, \dots, v_{n+1}$ be vectors in a vector space $V$. Suppose that $v_{n+1} \in \mathrm{Span}(\{ v_1, \dots, v_n \})$. Then*

$$\mathrm{Span}(\{ v_1, \dots, v_n \}) = \mathrm{Span}(\{ v_1, \dots, v_{n+1} \}).$$

# Efficiency

## Definition

A subset $S$ of a vector space $V$ is called linearly dependent if there are finitely many distinct vectors $u_1, \ldots, u_n$ in $S$ and scalars $a_1, \ldots, a_n$ not all equal to 0 such that

$$a_1 u_1 + \cdots + a_n u_n = 0_V. \tag{‡}$$

## Remark

The words "distinct" and "not all equal to 0" are critical. If $a_k = 0$ for all $k$, then (‡) is always satisfied no matter what the $u_k$ are. We call this a trivial representation of $0_V$. If $u_1 = u_2$, then $-(1)u_1 + (1)u_2 = 0_V$ would be a nontrivial representation of $0_V$ which would tell us nothing about the set $S$.

# Example

## Example

Let $A_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $A_2 = \begin{pmatrix} 0 & -1 \\ 3 & 0 \end{pmatrix}$, $A_3 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, and $A_4 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and consider $S = \{ A_1, A_2, A_3, A_4 \} \subset M_{2\times 2}(\mathbf{R})$. To see if $S$ is linearly dependent, we have to look for a nontrivial solution to $a_1A_1 + a_2A_2 + a_3A_3 + a_4A_4 = O$. This results in the system

$$\begin{aligned} a_1 + \phantom{a_2 + {}} a_3 + a_4 &= 0 \\ a_1 - a_2 + a_3 \phantom{{} + a_4} &= 0 \\ a_1 + 3a_2 + \phantom{a_3 + {}} a_4 &= 0 \\ a_1 + \phantom{a_2 + {}} a_3 + a_4 &= 0. \end{aligned}$$

You can confirm that $a_1 = 2$, $a_2 = -1$, $a_3 = -3$, and $a_4 = 1$ is a nontrivial solution. Hence $S$ is linearly dependent. (In fact, all the solutions are $\{ (2t, -t, -3t, t) : t \in \mathbf{R} \}$ and it might be a good idea to check this.)

# Redundancy

## Remark

Notice that in the process of showing that the set $S$ on the previous slide was linearly dependent, we discovered some relationships between the $A_k$. For example, $A_2 = 2A_1 - 3A_3 + A_4$ as well as $A_1 = \frac{1}{2}A_2 + \frac{3}{2}A_3 - \frac{1}{2}A_4$, etc. So for example, $\text{Span}\{ A_1, A_2, A_3, A_4 \} = \text{Span}\{ A_1, A_3, A_4 \}$ using our result from last time.

## Proposition

*Let $S$ be a subset of a vector space $V$. Then $S$ is linearly dependent if and only if (at least) one of the vectors in $S$ can be written as a linear combination of other vectors from $S$.* return

## Proof.

Suppose $v \in S$ is such that $v = a_1 u_1 + \cdots + a_n u_n$ for distinct vectors $u_1, \ldots, u_n$ is $S$ not equal to $v$. Then $0_V = a_1 u_1 + \cdots + a_n u_n + (-1)v$ is a nontrivial representation of $0_V$ and $S$ is linearly dependent.

# Proof

## Proof Continued.

Conversely, suppose that $S$ is linearly dependent. Then there are distinct vectors $u_1, \dots, u_n$ is $S$ and scalars $a_1, \dots, a_n$ not all equal to 0 so that $a_1 u_1 + \cdots + a_n u_n = 0_V$. Renumbering the vectors if necessary, we can assume $a_n \neq 0$. Then we have

$$u_n = -\frac{a_1}{a_n} u_1 - \cdots - \frac{a_{n-1}}{a_n} u_{n-1}.$$

That is, $u_n$ is a linear combination of $u_1, \dots, u_{n-1}$ from $S$. □

# Linear Independence

**Remark**

If we want to find a spanning set for $V$, then a linearly dependent set $S$ is guaranteed to have redundancies—one of the vectors in $S$ can be written as a linear combination of the other vectors in $S$. So it makes sense to make the following (equally important) definition.

**Definition**

We call a subset $S$ in a vector space $V$ linearly independent if it is not linearly dependent.

# Low Hanging Fruit

## Remark

1. The empty set is linearly independent since a linearly dependent set can't be empty.
2. A set consisting of a single nonzero vector is linearly independent. If $v \neq 0_V$ and $av = 0_V$ then $a = 0$.
3. Let $S = \{ u, v \}$ be distinct vectors in a vector space $V$. Then $S$ is linearly independent if and only if neither $u$ nor $v$ is a multiple of the other. (This homework problem §1.5, #9.)
4. A set $S$ is linearly independent if and only if the only representations of $0_V$ as linear combinations of vectors from $S$ are the trivial ones. That is, if we assume $u_1, \dots, u_n$ are distinct vectors from $S$ such that there are scalars $a_1, \dots, a_n$ such that $a_1 u_1 + \cdots + a_n u_n = 0_V$, then $a_1 = \cdots = a_n = 0$.

# An Example

## Example

Let $V = \mathbf{R}^4$. Let $u_1 = (1, 1, 1, 1)$, $u_2 = (1, 1, 1, 0)$, $u_3 = (1, 1, 0, 0)$ and $u_4 = (1, 0, 0, 0)$. Is $S = \{ u_1, u_2, u_3, u_4 \}$ linearly independent?

## Solution

*We need to show that* $a_1u_1 + a_2u_2 + a_3u_3 + a_4u_4 = (0, 0, 0, 0)$ *has only the trivial solution. But this gives us the system*

$$\begin{aligned} a_1 + a_2 + a_3 + a_4 &= 0 \\ a_1 + a_2 + a_3 \phantom{{}+a_4} &= 0 \\ a_1 + a_2 \phantom{{}+a_3+a_4} &= 0 \\ a_1 \phantom{{}+a_2+a_3+a_4} &= 0. \end{aligned}$$

*Since the only solution is* $a_1 = a_2 = a_3 = a_4 = 0$*, the set is linearly independent.*

# Break Time

Time for a short break and some questions.

# Linear Independence

**Theorem**

*Let $V$ be a vector space. Suppose that $S_1$ and $S_2$ are subsets such that $S_1 \subset S_2$.*

1. *If $S_1$ is linearly dependent, then so is $S_2$.*
2. *If $S_2$ is linearly independent, then so is $S_1$.*

**Proof.**

Item 2 follows from item 1. Why?
So we just need to prove item 1. Suppose $S_1$ is linearly dependent. Then there are distinct vectors $u_1, \dots, u_n$ and scalars $a_1, \dots, a_n$, not all equal to 0, such that

$$0_V = a_1 u_1 + \cdots + a_n u_n \tag{1}$$

is a nontrivial representation of $0_V$ from vectors in $S_1$ But the $u_k$ are also in $S_2$. Hence (1) is also a nontrivial representation of $0_V$ from vectors in $S_2$. This shows that $S_2$ is linearly dependent. □

# A Theorem

## Theorem

*Suppose that $S$ is a linearly independent set in a vector space $V$. If $v \in V$, then $S' = \{v\} \cup S$ is linearly independent if and only if $v \notin \mathrm{Span}(S)$.*

## Proof.

Suppose that $S'$ is linearly independent. If we had $v \in \mathrm{Span}(S)$, then $v$ would be a linear combination of vectors from $S$. Then by our previous propostion, $S'$ would be linearly dependent. Hence $v \notin \mathrm{Span}(S)$.

# Proof

**Proof Continued.**

Conversely, suppose $v \notin \text{Span}(S)$. Let $u_1, \dots, u_n$ be distinct vectors from $S' = \{v\} \cup S$ and $a_k$ scalars such that $a_1 u_1 + \cdots + a_n u_n = 0_V$. If none of the $u_k = v$, then all the $a_k = 0$ since $S$ is linearly independent. So we may as well assume that $u_1 = v$ and that $a_1 \neq 0$. But then

$$v = -\frac{a_2}{a_1} u_2 - \cdots - \frac{a_n}{a_1} u_n.$$

But $u_2, \dots, u_n$ are all in $S$. Then $v \in \text{Span}(S)$ and we get a contradiction. □

# Break Time

Time for a break and some questions.

# Bases

## Definition

A basis for a vector space $V$ is a linearly independent set that generates $V$.

## Example

Since $\emptyset$ is linearly independent and since we defined $\operatorname{Span}(\emptyset) = \{0_V\}$, the empty set is a basis for the trivial vector space $\{0_V\}$.

## Example (Standard Basis)

Let $V = \mathbf{F}^n$. Let $e_1 = (1, 0, \dots, 0)$, $e_2 = (0, 1, 0, \dots, 0)$,. . . ,$e_n = (0, \dots, 0, 1)$. Then $\{\, e_1, \dots, e_n \,\}$ is easily seen to be a basis for $\mathbf{F}^n$ usually called the standard basis for $\mathbf{F}^n$.

# More Examples

**Example (Standard Basis for $P_n(\mathbf{F})$)**

Let $V = P_n(\mathbf{F})$. Then $\{ 1, x, x^2, \dots, x^n \}$ is a basis for $P_n(\mathbf{F})$ called the standard basis for $P_n(\mathbf{F})$.

**Example**

Let $V = M_{m\times n}(\mathbf{F})$. Let $E^{ij} \in M_{m\times n}(\mathbf{F})$ be the matrix all of whose entries are 0 except for the $(i,j)^{\mathrm{th}}$ entry which is 1. Then $\{ E_{ij} : 1 \leq i \leq m \text{ and } 1 \leq j \leq n \}$ is a basis for $M_{m\times n}(\mathbf{F})$. For example if $m = n = 2$, then $\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \}$ is a basis for $M_{2\times 2}(\mathbf{F})$.

# Lots of Bases

## Example

Let $S = \{ u_1, u_2, u_3, u_4 \} \subset \mathbf{R}^4$ where $u_1 = (1,1,1,1)$, $u_2 = (1,1,1,0)$, $u_3 = (1,1,0,0)$ and $u_4 = (1,0,0,0)$. We already showed $S$ is linearly independent. To see that it is a basis for $\mathbf{R}^4$ we have to see that it spans $\mathbf{R}^4$. That is, we need to know that
$a_1 u_1 + a_2 u_2 + a_3 u_3 + a_4 u_4 = (x,y,z,w)$ always has a solution. That means we have to solve

$$\begin{aligned} a_1 + a_2 + a_3 + a_4 &= x \\ a_1 + a_2 + a_3 \qquad &= y \\ a_1 + a_2 \qquad\qquad &= z \\ a_1 \qquad\qquad\qquad &= w. \end{aligned}$$

But this is straightforward: $a_1 = w$, $a_2 = z - w$, $a_3 = y - w - z + w = y - z$, and $a_4 = x - w - (z - w) - (y - z) = x - y$. Hence $S = \{ u_1, u_2, u_3, u_4 \}$ is a basis for $\mathbf{R}^4$.

# Bases are Cool

## Theorem

*Let $V$ be a vector space and $\{ u_1, \dots, u_n \}$ vectors in $V$. Then $\{ u_1, \dots, u_n \}$ is a basis for $V$ if and only if every $v \in V$ can be written as a unique linear combination of the vectors $u_1, \dots, u_n$—that is, there are unique scalars $a_1, \dots, a_n$ such that $v = a_1 u_1 + \cdots + a_n u_n$.*

## Proof.

Suppose that $\{ u_1, \dots, u_n \}$ is a basis. If $v \in V$, then since $\{ u_1, \dots, u_n \}$ generates $V$, there are scalars $a_k$ such that $v = a_1 u_1 + \cdots + a_n u_n$. If we also have $v = b_1 u_1 + \cdots + b_n u_n$ for scalars $b_k$, then $0_V = (a_1 - b_1)u_1 + \cdots + (a_n - b_n)u_n$. Since $\{ u_1, \dots, u_n \}$ is linearly independent, this can only happen if $a_k - b_k = 0$ for all $k$. That is, we must have $a_k = b_k$ for all $k$. This proves the first part of the result.

# Proof

**Proof Continued.**

Now suppose every $v \in V$ is a unique linear combination of the $u_k$. Then by assumption $\{ u_1, \dots, u_n \}$ spans $V$. But if $0_V = a_1 u_1 + \cdots + a_n u_n$, then the $a_k$ must all be zero by uniqueness since $0_V = 0 \cdot u_1 + \cdots + 0 \cdot u_n$. Thus $\{ u_1, \dots, u_n \}$ is a basis. □

# Enough

1. That is enough for today.