# Math 24: Winter 2021
# Lecture 6

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# Let's Get Started

1. We should be recording.
2. Remember, it is more comfortable for me if you turn on your video so that I feel like I am talking to real people.
3. But first, are there any questions from last time?

# Review

## Definition

A subset $S$ of a vector space $V$ is called linear dependent if there are finitely many distinct vectors $u_1, \ldots, u_n$ in $S$ and scalars $a_1, \ldots, a_n$ not all equal to 0 such that

$$a_1 u_1 + \cdots + a_n u_n = 0_V.$$

A set which is not linearly dependent is called linearly independent.

## Proposition

*Let $S$ be a subset of a vector space $V$. Then $S$ is linearly dependent if and only if (at least) one of the vectors in $S$ can be written as a linear combination of other vectors from $S$.*

# More Review

**Theorem**

*Suppose that $S$ is a linearly independent set in a vector space $V$. If $v \in V$, then $S' = \{v\} \cup S$ is linearly independent if and only if $v \notin \mathrm{Span}(S)$.* return

**Definition**

A basis for a vector space $V$ is a linearly independent set that generates $V$.

**Theorem**

*Let $V$ be a vector space and $\{ u_1, \dots, u_n \}$ vectors in $V$. Then $\{ u_1, \dots, u_n \}$ is a basis for $V$ if and only if every $v \in V$ can be written as a unique linear combination of the vectors $u_1, \dots, u_n$—that is, there are unique scalars $a_1, \dots, a_n$ such that $v = a_1 u_1 + \cdots + a_n u_n$.*

# Finite Please

**Remark**

We saw last time, that $\{1, x, x^2, \ldots, x^n\}$ is a basis for $P_n(\mathbf{F})$. In particular, for any $n$, $\{1, x, x^2, \ldots, x^n\}$ is linearly independent in $P_n(\mathbf{F})$ and therefore in $P(\mathbf{F})$. Therefore the infinite set $S = \{1, x, x^2, \ldots\}$ clearly generates $P(\mathbf{F})$ and is linearly independent. Therefore it is a basis for $P(\mathbf{F})$. We will see shortly that this means $P(\mathbf{F})$ does not have a finite basis. We will work almost exclusively with vector spaces that have finite bases!

**Theorem**

*Suppose that $V$ is a vector space and that $S$ is a finite subset of $V$ that generates $V$. Then some subset of $S$ is a basis for $V$ and $V$ has a finite basis.*

# Proof

**Proof.**

If $S = \emptyset$, then $V = \text{Span}(\emptyset) = \{0_V\}$ and $S$ is a basis for $V$. Otherwise, there is a nonzero vector $u_1 \in S$. Note that $\{u_1\}$ is linearly independent. Since $S$ is finie, we can continue adding vectors $u_2$, $u_3$, etc., from $S$ such that $\{ u_1, \dots, u_n \}$ is linearly independent but no larger subset of $S$ is linearly independent.

If $\{ u_1, \dots, u_n \} = S$, then $S$ itself is a finite basis.

I claim that $\beta = \{ u_1, \dots, u_n \}$ is a basis for $V$. Since we know that $\beta$ is linearly independent by construction, we just have to show that $\text{Span}(\beta) = V$. For this, it suffices to see that $S \subset \text{Span}(\beta)$. Let $v \in S$. If $v \in \beta$, then clearly $v \in \text{Span}(\beta)$. If $v \notin \beta$, then $\{v\} \cup \beta$ is not linearly independent by our construction of $\beta$. Hence we must have $v \in \text{Span}(\beta)$ by our ▸ theorem from last time. Hence $S \subset \text{Span}(\beta)$ and $V \subset \text{Span}(S) \subset \text{Span}(\beta)$. Thus $\beta \subset S$ is a finite basis for $V$. □

# Break Time

Time for a short break and some questions.

# An Aisde: Induction

## Remark (A Simple Observation)

Let $A$ be a subset of $\mathbf{N} = \{1, 2, 3, \dots\}$. Suppose that $1 \in A$ and if $n \in A$, then $n + 1 \in A$. Then $A = \mathbf{N}$. We can use this basic observation to prove things. Let $P(n)$ be a statement that depends on $n$ and is either true of false. Suppose we prove that $P(1)$ is true and that whenever $P(n)$ is true for some $n \geq 1$, then $P(n + 1)$ is true. Then the set $A = \{n \in \mathbf{N} : P(n) \text{ is true}\}$ has exactly the above property and must be all of $\mathbf{N}$!

## Remark (Proof by Induction)

Thus to prove a statement $P(n)$ is true for all $n \in \mathbf{N}$, we can proceed as follows.

1. Prove that $P(1)$ is true.
2. Assume that $P(n)$ is true for some $n \geq 1$. (This is called the Inductive Hypothesis.
3. Then use the truth of $P(n)$ to prove that $P(n + 1)$ is true.

# Example

## Example

Show that $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$.

## Solution

*Here $P(n)$ is the statement that $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$. This is clearly true if $n = 1$. Suppose $P(n)$ holds for some $n \geq 1$. Then we have*

$$\begin{aligned} 1 + 2 + \cdots + n + 1 &= (1 + 2 + \cdots + n) + (n+1) \\ &= \frac{n(n+1)}{2} + n + 1 = \frac{1}{2}(n^2 + n + 2n + 2) \\ &= \frac{(n+1)(n+2)}{2} = \frac{(n+1)((n+1)+1)}{2}. \end{aligned}$$

*Therefore $P(n+1)$ holds. This completes the proof.*

# Replacement Theorem

**Theorem (Replacement Theorem)**

*Let $V$ be a vector space. Suppose that $G$ is a finite set of $n$ elements that generates $V$. Let $L$ be a linearly independent subset of $V$ containing $m$ vectors. Then $m \leq n$ and there is a subset $H \subset G$ containing exactly $n - m$ vectors such that $L \cup H$ generates $V$.*

**Proof.**

If $L = \emptyset$, then $m = 0$ and we can let $H = G$.

If $m = 1$, then $L = \{v\}$ for a nonzero vector $v$. Since $G = \{ u_1, \dots, u_n \}$ generates, $v = a_1 u_1 + \cdots + a_n u_n$. Since $v \neq 0_V$, at least one $a_k \neq 0$. We may as well assume $a_1 \neq 0$. Then

$$u_1 = \frac{1}{a_1} v - \frac{a_2}{a_1} u_2 - \cdots - \frac{a_n}{a_1} u_n,$$

# Proof

## Proof Continued.

Then I claim we can let $H = \{ u_2, \dots, u_n \}$. Since $H$ has $n-1$ elements, it suffices to see that it generates. But it follows from the last slide that
$\{ u_1, \dots, u_n \} \subset \mathrm{Span}(L \cup H)$. Then
$V = \mathrm{Span}(\{ u_1, \dots, u_n \} \subset \mathrm{Span}(L \cup H)$ and we've proven the result when $m = 1$.

So we proceed by induction and assume that result when $L$ has $m$ elements for some $m \geq 1$.

Now let $L = \{ v_1, \dots, v_m, v_{m+1} \}$. Then $L' = \{ v_1, \dots, v_m \}$ is linearly independent with $m$ elements. Then by the Induction hypothesis, $m \leq n$ and there is a subset $H' = \{ u_1, \dots, u_{n-m} \} \subset G$ containing $n - m$ elements such that $L' \cup H'$ generates $V$.

# Proof

## Proof Continued.

Therefore, for appropriate scalars,

$$v_{m+1} = a_1 v_1 + \cdots + a_m v_m + b_1 u_1 + \cdots + b_{n-m} u_{n-m}.$$

Since $L$ is linearly independent, $v \notin \operatorname{Span}(L')$ and we must have some $b_k \neq 0$. We may as well assume $b_1 \neq 0$. Then

$$u_1 = -\frac{a_1}{b_1} v_1 - \cdots - \frac{a_m}{b_1} v_m + \frac{1}{b_1} v_{m+1} - \frac{b_2}{a_1} u_2 - \cdots - \frac{b_{n-m}}{b_1} u_{n-m}.$$

# Proof

**Proof Continued.**

Therefore if we let $H = \{ u_2, \ldots, u_{n-m} \}$, then $H$ has $n - (m+1)$ elements and the previous slide shows

$$u_1 \in \operatorname{Span}(L \cup H).$$

Since $L' \subset L$ and $H' = \{u_1\} \cup H$, we have $L' \cup H' \subset \operatorname{Span}(L \cup U)$ Then by assumption

$$V = \operatorname{Span}(L' \cup H') \subset \operatorname{Span}(L \cup U).$$

Hence $L \cup H$ generates $V$. This completes the proof. □

# The Pay Off

**Corollary**

*Suppose that $V$ is a vector space with a finite basis. Then every basis of $V$ is finite and has the same number of vectors.*

**Proof.**

Let $\beta = \{ u_1, \dots, u_n \}$ be a finite basis for $V$. Suppose that $\alpha$ is another basis. If $\alpha$ were infinite, it would contain a linearly independent subset $L$ with $n+1$ vectors. This contradicts our theorem since $\beta$ generates $V$. Hence $\alpha = \{ v_1, \dots, v_m \}$ must be finite. Since $\alpha$ is linearly independent and $\beta$ generates, $m \leq n$. Now we can reverse the roles of $\alpha$ and $\beta$ to see that $n \leq m$. $\square$

# Dimension

## Definition

A vector space is called finite dimensional if it has a finite basis. The common number of elements in any finite basis is called the dimension of $V$ and is denoted by $\dim(V)$. If $V$ does not have a finite basis, then we say that $V$ is infinite dimensional.

## Example

1. The dimension of the trivial space $\{0_V\}$ is zero.
2. We have $\dim(\mathbf{F}^n) = n$.
3. We have $\dim\big(\mathrm{P}_n(F)\big) = n + 1$.
4. We have $\dim\big(M_{m\times n}(\mathbf{F})\big) = mn$.
5. The vector space $\mathrm{P}(\mathbf{F})$ is infinite dimensional.

# Enough

1. That is enough for today.