Math 24: Winter 2021
Lecture 6

Dana P. Williams

Dartmouth College

Thursday, January 21, 2021

Let's Get Started

1. We should be recording.
2. Remember, it is more comfortable for me if you turn on your video so that I feel like I am talking to real people.
3. But first, are there any questions from last time?

Review

Definition

A subset S of a vector space V is called linearly dependent if there are finitely many distinct vectors $u_1, \ldots, u_n$ in S and scalars $a_1, \ldots, a_n$ not all equal to 0 such that

$$a_1 u_1 + \cdots + a_n u_n = 0_V.$$

A set which is not linearly dependent is called linearly independent.

Proposition

Let S be a subset of a vector space V. Then S is linearly dependent if and only if (at least) one of the vectors in S can be written as a linear combination of other vectors from S.

More Review

Theorem

Suppose that S is a linearly independent set in a vector space V. If $v \in V$, then $S' = \{v\} \cup S$ is linearly independent if and only if $v \notin \operatorname{Span}(S)$. return

Definition

A basis for a vector space V is a linearly independent set that generates V.

Theorem

Let V be a vector space and $\{ u_1, \dots, u_n \}$ vectors in V. Then $\{ u_1, \dots, u_n \}$ is a basis for V if and only if every $v \in V$ can be written as a unique linear combination of the vectors $u_1, \dots, u_n$—that is, there are unique scalars $a_1, \dots, a_n$ such that $v = a_1 u_1 + \cdots + a_n u_n$.

Finite Please

Remark

We saw last time, that $\{1, x, x^2, \ldots, x^n\}$ is a basis for $\mathrm{P}_n(\mathbf{F})$. In particular, for any n, $\{1, x, x^2, \ldots, x^n\}$ is linearly independent in $\mathrm{P}_n(\mathbf{F})$ and therefore in $\mathrm{P}(\mathbf{F})$. Therefore the infinite set $S = \{1, x, x^2, \ldots\}$ clearly generates $\mathrm{P}(\mathbf{F})$ and is linearly independent. Therefore it is a basis for $\mathrm{P}(\mathbf{F})$. We will see shortly that this means $\mathrm{P}(\mathbf{F})$ does not have a finite basis. We will work almost exclusively with vector spaces that have finite bases!

Theorem

Suppose that V is a vector space and that S is a finite subset of V that generates V. Then some subset of S is a basis for V and V has a finite basis.

Proof

Proof.

If $S = \emptyset$, then $V = \text{Span}(\emptyset) = \{0_V\}$ and S is a basis for V. Otherwise, there is a nonzero vector $u_1 \in S$. Note that $\{u_1\}$ is linearly independent. Since S is finite, we can continue adding vectors u_2, u_3, etc., from S until $\{ u_1, \dots, u_n \}$ is linearly independent but no larger subset of S is linearly independent.

If $\{ u_1, \dots, u_n \} = S$, then S itself is a finite basis.

Otherwise, I claim that $\beta = \{ u_1, \dots, u_n \}$ is a basis for V. Since β is linearly independent by construction, we just have to show that $\text{Span}(\beta) = V$. For this, it suffices to see that $S \subset \text{Span}(\beta)$. Let $v \in S$. If $v \in \beta$, then clearly $v \in \text{Span}(\beta)$. If $v \notin \beta$, then $\{v\} \cup \beta$ is not linearly independent by our construction of β. Hence we must have $v \in \text{Span}(\beta)$ by our ▸ theorem from last time. Hence $S \subset \text{Span}(\beta)$ and $V = \text{Span}(S) \subset \text{Span}(\beta)$. Thus $\beta \subset S$ is a finite basis for V. □

Break Time

Time for a short break and some questions.

An Aisde: Induction

Remark (A Simple Observation)

Let A be a subset of $\mathbf{N} = \{1, 2, 3, \dots\}$. Suppose that $1 \in A$ and if $n \in A$, then $n + 1 \in A$. Then $A = \mathbf{N}$. We can use this basic observation to prove things. Let $P(n)$ be a statement that depends on n and is either true of false. Suppose we prove that $P(1)$ is true and that whenever $P(n)$ is true for some $n \geq 1$, then $P(n + 1)$ is true. Then the set $A = \{ n \in \mathbf{N} : P(n) \text{ is true} \}$ has exactly the above property and must be all of $\mathbf{N}$!

Remark (Proof by Induction)

Thus to prove a statement $P(n)$ is true for all $n \in \mathbf{N}$, we can proceed as follows.

1. Prove that $P(1)$ is true.
2. Assume that $P(n)$ is true for some $n \geq 1$. (This is called the Inductive Hypothesis.
3. Then use the truth of $P(n)$ to prove that $P(n + 1)$ is true.

Example

Example

Show that $1 + 2 + \cdots + n = \dfrac{n(n+1)}{2}$.

Solution

Here $P(n)$ is the statement that $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$. This is clearly true if $n = 1$. Suppose $P(n)$ holds for some $n \geq 1$. Then we have

$$\begin{aligned} 1 + 2 + \cdots + n + 1 &= (1 + 2 + \cdots + n) + (n+1) \\ &= \frac{n(n+1)}{2} + n + 1 = \frac{1}{2}(n^2 + n + 2n + 2) \\ &= \frac{(n+1)(n+2)}{2} = \frac{(n+1)((n+1)+1)}{2}. \end{aligned}$$

Therefore $P(n+1)$ holds. This completes the proof.

Replacement Theorem

Theorem (Replacement Theorem)

Let V be a vector space. Suppose that G is a finite set of n elements that generates V. Let L be a linearly independent subset of V containing m vectors. Then $m \leq n$ and there is a subset $H \subset G$ containing exactly $n - m$ vectors such that $L \cup H$ generates V.

Proof.

If $L = \emptyset$, then $m = 0$ and we can let $H = G$.

If $m = 1$, then $L = \{v\}$ for a nonzero vector v. Since $G = \{ u_1, \dots, u_n \}$ generates, $v = a_1 u_1 + \cdots + a_n u_n$. Since $v \neq 0_V$, at least one $a_k \neq 0$. We may as well assume $a_1 \neq 0$. Then

$$u_1 = \frac{1}{a_1} v - \frac{a_2}{a_1} u_2 - \cdots - \frac{a_n}{a_1} u_n,$$

Proof Continued.

Then I claim we can let $H = \{ u_2, \dots, u_n \}$. Since H has $n-1$ elements, it suffices to see that $L \cup H$ generates. But it follows from the last slide that
$\{ u_1, \dots, u_n \} \subset \operatorname{Span}(L \cup H)$. Then
$V = \operatorname{Span}(\{ u_1, \dots, u_n \} \subset \operatorname{Span}(L \cup H)$ and we've proven the result when $m = 1$.

So we proceed by induction and assume that result when L has m elements for some $m \geq 1$.

Now let $L = \{ v_1, \dots, v_m, v_{m+1} \}$. Then $L' = \{ v_1, \dots, v_m \}$ is linearly independent with m elements. Then by the Induction hypothesis, $m \leq n$ and there is a subset $H' = \{ u_1, \dots, u_{n-m} \} \subset G$ containing $n - m$ elements such that $L' \cup H'$ generates V.

Proof

Proof Continued.

Therefore, for appropriate scalars,

$$v_{m+1} = a_1 v_1 + \cdots + a_m v_m + b_1 u_1 + \cdots + b_{n-m} u_{n-m}.$$

Since L is linearly independent, $v_{m+1} \notin \text{Span}(L')$ and we must have some $b_k \neq 0$. We may as well assume $b_1 \neq 0$. Then

$$u_1 = -\frac{a_1}{b_1} v_1 - \cdots - \frac{a_m}{b_1} v_m + \frac{1}{b_1} v_{m+1} - \frac{b_2}{a_1} u_2 - \cdots - \frac{b_{n-m}}{b_1} u_{n-m}.$$

Proof

Proof Continued.

Therefore if we let $H = \{ u_2, \dots, u_{n-m} \}$, then H has $n - (m+1)$ elements and the previous slide shows

$$u_1 \in \operatorname{Span}(L \cup H).$$

Since $L' \subset L$ and $H' = \{u_1\} \cup H$, we have $L' \cup H' \subset \operatorname{Span}(L \cup H)$ Then by assumption

$$V = \operatorname{Span}(L' \cup H') \subset \operatorname{Span}(L \cup H).$$

Hence $L \cup H$ generates V. Since $L \cup H$ has exactly n vectors in it, we must have $m + 1 \leq n$. This completes the proof. □

Break Time

That was a big theorem! Let's take a break and see if there are any questions.

The Pay Off

Corollary

Suppose that V is a vector space with a finite basis. Then every basis of V is finite and has the same number of vectors.

Proof.

Let $\beta = \{ u_1, \dots, u_n \}$ be a finite basis for V. Suppose that α is another basis. If α were infinite, it would contain a linearly independent subset L with $n + 1$ vectors. This contradicts our theorem since β generates V. Hence $\alpha = \{ v_1, \dots, v_m \}$ must be finite. Since α is linearly independent and β generates, $m \leq n$. Now we can reverse the roles of α and β to see that $n \leq m$. □

Dimension

Definition

A vector space is called finite dimensional if it has a finite basis. The common number of elements in any finite basis is called the dimension of V and is denoted by $\dim(V)$. If V does not have a finite basis, then we say that V is infinite dimensional.

Example

1. The dimension of the trivial space $\{0_V\}$ is zero.
2. We have $\dim(\mathbf{F}^n) = n$.
3. We have $\dim\big(\mathrm{P}_n(F)\big) = n + 1$.
4. We have $\dim\big(M_{m\times n}(\mathbf{F})\big) = mn$.
5. The vector space $\mathrm{P}(\mathbf{F})$ is infinite dimensional.

Enough

1. That is enough for today.