Math 24: Winter 2021
Lecture 7

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Let's Get Started

1. We should be recording.
2. Remember, it is more comfortable for me if you turn on your video so that I feel like I am talking to real people.
3. Our preliminary exam will be available from Thursday at 11am (after office hours) and must be submitted by Saturday at 10pm EST. You will have 150 minutes to work on the exam with an extra 30 minutes for scanning and submitting via `gradescope`. The exam will cover through and including §2.2 in the text (which I hope to finish on Monday).
4. But first, are there any questions from last time?

Review

Theorem (Replacement Theorem)

Let V be a vector space. Suppose that G is a finite set of n elements that generates V. Let L be a linearly independent subset of V containing m vectors. Then $m \leq n$ and there is a subset $H \subset G$ containing exactly $n - m$ vectors such that $L \cup H$ generates V.

Corollary

Suppose that V is a vector space with a finite basis. Then every basis of V is finite and has the same number of vectors.

Definition

A vector space is called finite dimensional if it has a finite basis. The common number of elements in any finite basis is called the dimension of V and is denoted by $\dim(V)$. If V does not have a finite basis, then we say that V is infinite dimensional.

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Example

1. The dimension of the trivial space $\{0_V\}$ is zero.
2. We have $\dim(\mathbf{F}^n) = n$.
3. We have $\dim\big(\mathrm{P}_n(F)\big) = n + 1$.
4. We have $\dim\big(M_{m\times n}(\mathbf{F})\big) = mn$.
5. The vector space $\mathrm{P}(\mathbf{F})$ is infinite dimensional.

Given Dimension

Theorem (Corollary of the Replacement Theorem)

Suppose that V is a vector space with dimension n.

1. *Every generating set for V has at least n vectors and any generating set with n vectors is a basis.*
2. *Every linearly independent subset of V contains at most n vectors and a linearly independent set with n vectors is a basis.*
3. *Every linearly independent subset of V can be extended to a basis for V. (That is, if L is linearly independent, then there is a basis β for V such that $L \subset \beta$.)*

Proof

Proof.

Let β be a basis for V containing n elements.

(1) Suppose that G generates V. Then some subset H of G is a basis and must have n elements in it. Thus G has at least n elements. If G has exactly n elements, then $G = H$ and is a basis for V.

(2) If L is linearly independent and has m vectors in it, then $m \leq n$ by the Replacement Theorem and there is a subset H of β with $n - m$ elements such that $L \cup H$ generates. Thus if $m = n$, then $H = \emptyset$ and L also generates. Hence L is a basis.

(3) If L is linearly independent and $H \subset \beta$ is as in item 2 with $n - m$ elements, then $L \cup H$ is a generating set with exactly n elements. Hence $L \cup H$ must be a basis extending L by part (1). $\square$

Examples

Example

We saw in lecture 5 that $\beta = \{ (1,1,1,1), (1,1,1,0), (1,1,0,0), (1,0,0,0) \}$ is linearly independent in $\mathbf{R}^4$. Since $\dim(\mathbf{R}^4)) = 4$, β is a basis for $\mathbf{R}^4$. Hence it automatically generates $\mathbf{R}^4$ as we also proved directly in lecture 5.

Example

Let $p_1(x) = 1 - x + 2x^2 - x^3$, $p_2(x) = 7x + 3x^3$, and $p_3(x) = -2 + x^2 - 10x^3$. Since $\dim(P_3(\mathbf{R})) = 4$, $S = \{ p_1, p_2, p_3 \}$ does not generate $\mathrm{P}_3(\mathbf{R})$.

Subspaces

Theorem

Suppose that V is a finite-dimensional vector space. If W is a subspace of V, then W if finite dimensional and $\dim(W) \leq \dim(V)$. *If* $\dim(W) = \dim(V)$, *then* $W = V$.

Proof.

Let W be a subspace of V. If $W = \{0_V\}$ then W is finite dimensional with $\dim(W) = 0 \leq \dim(V)$. Otherwise, W contains a nonzero vector u_1 and $\{u_1\}$ is linearly independent. If $\mathrm{Span}(\{u_1\}) \neq W$, then we can find $u_2 \in W \setminus \mathrm{Span}(\{u_1\})$. Thus $\{ u_1, u_2 \} = \{u_1\} \cup \{u_2\}$ is linearly independent. Continuing, we get linearly independent sets $\{ u_1, \dots, u_m \}$ in W. Since $\{ u_1, \dots, u_m \}$ is linearly independent in V as well, $m \leq \dim(V)$.

Proof

Proof Continued.

Hence the process must terminate with some $m \leq \dim(V)$, and we get $\{ u_1, \dots, u_m \}$ linearly independent in W with $\mathrm{Span}(\{ u_1, \dots, u_m \}) = W$ and $m \leq \dim(V)$. Hence W is finite dimensional with $\dim(W) \leq \dim(V)$.

If $\dim(W) = \dim(V)$, then any basis β of W is a linearly independent set in V containing $\dim(V)$ vectors. Hence it must be a basis for V. Therefore $V = \mathrm{Span}(\beta) = W$. $\square$

A Corollary

Corollary

If W is a subspace of a finite dimensional vector space V, then any basis of W can be extended to a basis of V.

Proof.

A basis for W is a linearly independent set if V. Hence it can be extended to a basis. $\square$

Example

Recall that E^{ij} is the matrix with a 1 is the $(i,j)^{\mathrm{th}}$ slot and zeros elsewhere. Let W be the subspace of symmetric matrices in $M_{2\times 2}(\mathbf{R})$. We saw that $\beta = \{\, E^{11}, E^{22}, E^{12} + E^{21} \,\}$ generates W. In fact, $aE^{11} + bE^{22} + c(E^{12} + E^{21}) = \left(\begin{smallmatrix} a & c \\ c & d \end{smallmatrix}\right)$. Thus if $aE^{11} + bE^{22} + c(E^{12} + E^{21}) = O$, then $a = b = c = 0$. It follows that β is linearly independent and $\dim(W) = 3$. More generally, we can let A^{ij} be the $n \times n$-matrix with ones in the $(i,j)^{\mathrm{th}}$ and $(j,i)^{\mathrm{th}}$ slots. Thus $A^{ij} = E^{ij}$ if $i = j$ and $A^{ij} = E^{ij} + E^{ji}$ otherwise. Then it is not hard to see that $\beta = \{\, A^{ij} : 1 \le i \le j \le n \,\}$ is a basis for the subspace W of symmetric matrices in $M_{n\times n}(\mathbf{F})$. Then $\dim(W) = n + (n-1) + \cdots = 1 = \frac{n(n+1)}{2}$.

Break Time

Time for a well earned rest and perhaps some questions.

Definition

Let V and W be vector spaces over the same field $\mathbf{F}$. A function $T : V \to W$ is called a linear transformation or a linear map if for all $x, y \in V$ and $a \in \mathbf{F}$ we have

1. $T(x + y) = T(x) + T(y)$, and
2. $T(ax) = aT(x)$.

Remark

Linear transformations are very special sorts of functions. Our favorite functions from the Good ol' days of calculus are almost never linear. If $f(x) = x^2$, then outside of Enormous State University, we don't generally have $(x + y)^2 = f(x + y)$ equal to $f(x) + f(y) = x^2 + y^2$. In fact, if $T : \mathbf{R} \to \mathbf{R}$ is linear (viewing $\mathbf{R}$ as a one-dimensional real-vector space), then $T(x) = T(x \cdot 1) = xT(1)$. So linear transformations from $V = \mathbf{R}$ to itself are all of the form $T(x) = ax$ for some real number $a \in \mathbf{R}$. In fact, what we called linear functions back in the day, namely functions of the form $f(x) = mx + b$, are linear transformations only when $b = 0$!

Low Hanging Fruit

Proposition

Suppose that $T : V \to W$ is linear. Then $T(0_V) = 0_W$.

Proof.

$T(0_V) = T(0_V + 0_V) = T(0_V) + T(0_V)$. Now add $-T(0_V)$ to both sides. $\square$

Proposition

Suppose $T : V \to W$ is linear. If $v_1, \dots, v_n \in V$ and $a_k \in \mathbf{F}$, then

$$T\Big(\sum_{k=1}^{n} a_k v_k\Big) = \sum_{k=1}^{n} a_k T(v_k).$$

Proof.

We have
$T\Big(\sum_{k=1}^{n} a_k v_k\Big) = T\Big(a_1 v_1 + \sum_{k=2}^{n} a_k v_k\Big) = a_1 T(v_1) + T\Big(\sum_{k=2}^{n} a_k v_k\Big)$.
Now use induction. $\square$

Examples

Lemma

A function $T : V \to W$ is linear if and only if $T(ax + y) = aT(x) + T(y)$ for all $x, y \in V$ and $a \in \mathbf{F}$.

Proof.

This is an exercise. $\square$

Example

Define $T : \mathbf{R}^2 \to \mathbf{R}^2$ by $T(x, y) = \big(x - y, 2x + y\big)$. Show that T is linear.

Solution

We use the above lemma. We have $T\big(a(x, y) + (x', y')\big) = T(ax + x', ay + y') = (ax + x' - ay - y', 2ax + 2x' + ay + y') = a(x - y, 2x + y) + (x' - y', 2x' + y') = aT(x, y) + T(x', y')$. *Thus T is linear.*

More Examples

Example (The Transpose)

Let $V = M_{m\times n}(\mathbf{F})$ and $W = M_{n\times m}(\mathbf{F})$. Define $T : M_{m\times n}(\mathbf{F}) \to M_{n\times m}(\mathbf{F})$ by $T(A) = A^t$. Then you showed on homework that $T(aA + B) = (aA + B)^t = aA^t + B^t = aT(A) + T(B)$. Hence T is linear (by our lemma).

Example (Differentiation)

Let $V = C^\infty(\mathbf{R})$ be the subset of $C(\mathbf{R})$ consisting of functions which have derivatives of all orders at every point. (Such functions are sometimes called smooth.) Since the derivative of sum is the sum of the derivatives, it is not hard to verify that V is a subspace—and hence a real-vector space. We can define $T : V \to V$ by $T(f) = f'$. Then T is linear: $T(af + g) = (af + g)' = af' + g' = aT(f) + T(g)$.

Good Friends

Definition

If V is a vector space over $\mathbf{F}$, then the identity transformation is the map $I_V : V \to V$ given by $I_V(x) = x$ for all $x \in V$. If W is also a vector space over $\mathbf{F}$, then the zero transformation $T_0 : V \to W$ is given by $T_0(v) = 0_W$ for all $v \in V$.

Remark

The identity transformation and the zero transformation are easily seen to be linear transformations. When there is no ambiguity about V, we sometimes write I in place of I_V.

Break Time

Time to relax a bit and ask some questions.

The Null Space and Range

Definition

Suppose that $T : V \to W$ is a linear transformation. Then the null space or kernel of T is the set $\mathrm{N}(T) = \{\, v \in V : T(v) = 0_W \,\}$. The range of T is the set $\mathrm{R}(T) = \{\, T(v) : v \in V \,\}$.

Example

Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be given by $T(x, y) = (x - y, 0)$. Then it is an exercise to verify that T is linear, that
$\mathrm{N}(T) = \{\, (x, x) : x \in \mathbf{R} \,\} = \mathrm{Span}(\{(1, 1)\})$, and that
$\mathrm{R}(T) = \{\, (x, 0); x \in \mathbf{R} \,\} = \mathrm{Span}(\{(1, 0)\})$.

Subspaces

Proposition

If $T : V \to W$ is linear, then $\mathrm{N}(T)$ is a subspace of V and $\mathrm{R}(T)$ is a subspace of W.

Proof.

Since we always have $T(0_V) = 0_W$, we have $0_V \in \mathrm{N}(T)$. If $x, y \in \mathrm{N}(T)$ and $a \in \mathbf{F}$, then $T(ax + y) = aT(x) + T(y) = a0_W + 0_W = 0_W$. Hence $ax + y \in \mathrm{N}(T)$. Thus $\mathrm{N}(T)$ is a subspace of V.

Note that the above shows $0_W \in \mathrm{R}(T)$. If $u, v \in \mathrm{R}(T)$ and $a \in \mathbf{F}$, then, by definition, there are $x, y \in V$, such that $T(x) = u$ and $T(y) = v$. But then $ax + y \in V$, and $T(ax + y) = aT(x) + T(y) = au + v$. Hence $au + v \in \mathrm{R}(T)$ and $\mathrm{R}(T)$ is a subspace. $\square$

Spans

Proposition

Suppose that $T : V \to W$ is linear and that $\beta = \{ v_1, \dots, v_n \}$ is a basis for V. Then $\{ T(v_1), \dots, T(v_n) \}$ generates $\mathrm{R}(T)$. That is,

$$\mathrm{R}(T) = \mathrm{Span}(T(\beta)) = \mathrm{Span}(\{ T(v_1), \dots, T(v_n) \}).$$

Proof.

Since $T(v_k) \in \mathrm{R}(T)$ for all k and $\mathrm{R}(T)$ is a subspace, we have $\mathrm{Span}(T(\beta)) = \mathrm{Span}(\{ T(v_1), \dots, T(v_n) \}) \subset \mathrm{R}(T)$. For the other containment, consider $w \in \mathrm{R}(T)$. Then $w = T(v)$ for some $v \in V$. Since β is a basis for V, $v = a_1 v_1 + \cdots + a_n v_n$ for unique scalars a_k. But then $w = T(v) = a_1 T(v_1) + \cdots + a_n T(v_n)$ and $w \in \mathrm{Span}(\{ T(v_1), \dots, T(v_n) \})$. Since $w \in \mathrm{R}(T)$ was arbitrary, we're done. □

Enough

1. That is enough for today.