Math 24: Winter 2021
Lecture 8

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Monday, January 25, 2021

Let's Get Started

1. We should be recording.
2. Remember, it is more comfortable for me if you turn on your video so that I feel like I am talking to real people.
3. Our preliminary exam will be available from Thursday at 11am (after office hours) and must be submitted by Saturday at 10pm EST. You will have 150 minutes to work on the exam with an extra 30 minutes for scanning and submitting via `gradescope`. The exam will cover through and including §2.2 in the text (which I hope to finish Today).
4. I suggest blocking out a three hour window now so that you can work undisturbed. Although officially a take home exam, it is effectively an "in class" exam where you get to pick when you take it.
5. But first, are there any questions from last time?

Linear Maps

Definition

Let V and W be vector spaces over the same field $\mathbf{F}$. A function $T : V \to W$ is called a linear transformation or a linear map if for all $x, y \in V$ and $a \in \mathbf{F}$ we have

1. $T(x + y) = T(x) + T(y)$, and
2. $T(ax) = aT(x)$.

Proposition

Suppose that $T : V \to W$ is linear. Then $T(0_V) = 0_W$.

Proposition

Suppose $T : V \to W$ is linear. If $v_1, \dots, v_n \in V$ and $a_k \in \mathbf{F}$, then

$$T\Big(\sum_{k=1}^{n} a_k v_k\Big) = \sum_{k=1}^{n} a_k T(v_k).$$

The Null Space and Range

Definition

Suppose that $T : V \to W$ is a linear transformation. Then the null space or kernel of T is the set $\mathrm{N}(T) = \{ v \in V : T(v) = 0_W \}$. The range of T is the set $\mathrm{R}(T) = \{ T(v) : v \in V \}$.

Proposition

If $T : V \to W$ is linear, then $\mathrm{N}(T)$ is a subspace of V and $\mathrm{R}(T)$ is a subspace of W.

Rank and Nullity

Definition

Suppose that $T : V \to W$ is a linear transformation. If $\mathrm{N}(T)$ and $\mathrm{R}(T)$ are finite dimensional, then we call $\mathrm{nullity}(T) = \dim(\mathrm{N}(T))$ the nullity of T and $\mathrm{rank}(T) = \dim(\mathrm{R}(T))$ the rank of T.

Remark

One of the things that makes linear transformations special is that they "preserve dimension" as described in the next result. In our text, it is called the Dimension Theorem. Other texts call it the Rank-Nullity Theorem.

Theorem (Dimension Theorem)

Suppose that $T : V \to W$ is linear and V is finite dimensional. Then

$$\dim(V) = \mathrm{nullity}(T) + \mathrm{rank}(T).$$

Proof.

Let $n = \dim(V)$. Suppose that $\text{nullity}(T) = \dim(\text{N}(T)) = k$ and that $\{ v_1, \ldots, v_k \}$ is a basis for $\text{N}(T)$. Note that $0 \leq k \leq n$. Then we can extend $\{ v_1, \ldots, v_k \}$ to a basis $\beta = \{ v_1, \ldots, v_k, v_{k+1}, \ldots, v_n \}$ for V. Since $n = k + (n - k)$, it will suffice to prove that $\alpha = \{ T(v_{k+1}), \ldots, T(v_n) \}$ is a basis for $\text{R}(T)$. Since β generates V, $\text{R}(T) = \text{Span}(T(\beta)) = \text{Span}(\{ T(v_1), \ldots, T(v_n) \}) = \text{Span}(\{ T(v_{k+1}), \ldots, T(v_n) \})$. Hence α generates $\text{R}(T)$. Hence we just need to see that α is linearly independent.

Proof continued.

Suppose that $\sum_{j=k+1}^{n} b_j T(v_j) = 0_W$. We need to prove that this forces $b_j = 0$ for $k+1 \le j \le n$. Since T is linear, we have $T\big(\sum_{j=k+1}^{n} b_j v_j\big) = 0_W$ and $\sum_{j=k+1}^{n} b_j v_j \in \mathrm{N}(T)$. Hence there are scalars a_j such that

$$\sum_{j=k+1}^{n} b_j v_j = \sum_{j=1}^{k} a_j v_j.$$

Thus if $b_j = -a_j$ for $1 \le j \le k$, then

$$\sum_{j=1}^{n} b_j v_j = 0_V.$$

Since β is a basis, and hence linearly independent, this forces $b_j = 0$ for all $1 \le j \le n$. But then $b_j = 0$ for $k+1 \le j \le n$. $\square$

Review from Appendix B

Remark

Recall that a function $f : V \to W$ is called one-to-one if $f(v_1) = f(v_2)$ implies $v_1 = v_2$. We call a function $f : V \to W$ onto if given $w \in W$ there is a $v \in V$ such that $T(v) = w$. Thus if T is linear, then T is onto if and only if $\mathrm{R}(T) = W$.

Proposition

Suppose that $T : V \to W$ is a linear transformation. Then T is one-to-one if and only if $\mathrm{N}(T) = \{0_V\}$.

Proof

Proof.

Suppose T is one-to-one and $x \in \mathrm{N}(T)$. Then $T(x) = 0_W = T(0_V)$. Thus $x = 0_V$ and $\mathrm{N}(T) = \{0_V\}$.

Conversely, suppose that $\mathrm{N}(T) = \{0_V\}$. Suppose that $T(x_1) = T(x_2)$. Then since T is linear, $T(x_1 - x_2) = 0_W$. Thus $x_1 - x_2 \in \mathrm{N}(T)$. Hence $x_1 - x_2 = 0_V$ and $x_1 = x_2$. That is, T is one-to-one. □

Break Time

Time for a break and some questions.

One-To-One and Onto

Theorem

Suppose that V and W are finite-dimensional vector spaces over $\mathbf{F}$ with $\dim(V) = \dim(W)$. Suppose also that $T : V \to W$ is linear. Then the following are equivalent.

1. *T is one-to-one.*
2. $\operatorname{rank}(T) = \dim(V)$.
3. *T is onto.*

Proof.

Recall that the Dimension Theorem implies $\dim(V) = \operatorname{nullity}(T) + \operatorname{rank}(T)$.
(1)$\Longrightarrow$(2): If T is one-to-one, then $N(T) = \{0_V\}$ and $\operatorname{nullity}(T) = 0$. Thus $\operatorname{rank}(T) = \dim(V)$.
(2)$\Longrightarrow$(3): If $\operatorname{rank}(T) = \dim(V)$, then $\operatorname{rank}(T) = \dim(R(T)) = \dim(W)$. Hence $R(T) = W$ and T is onto.
(3)$\Longrightarrow$(1): If T is onto, then $R(T) = W$. Hence $\operatorname{rank}(T) = \dim(W) = \dim(V)$. Hence $\operatorname{nullity}(T) = 0$ and $N(T) = \{0_V\}$. Hence T is one-to-one. $\square$

The Importance of Being Finite Dimensional

Example

If $V = \mathrm{P}(\mathbf{R})$, then, since we are working over the reals, we can view polynomials as functions and $\mathrm{P}(\mathbf{R})$ is a subspace of $\mathscr{F}(\mathbf{R}, \mathbf{R})$. (Unless stated explicitly otherwise, this will be the normal state of affairs in Math 24!) Since the derivative of a polynomial is a polynomial, $T(p) = p'$ is a linear map $T : \mathrm{P}(\mathbf{R}) \to \mathrm{P}(\mathbf{R})$. It is clearly onto: given $p \in \mathrm{P}(\mathbf{F})$, let

$$q(x) = \int_0^x p(t)\,dt.$$

Then $T(q) = p$. But T is not one-to-one! $T(p_1) = T(p_2)$ if and only if p_1 and p_2 differ by a constant (polynomial).

Example

Example

Define $T : \mathrm{P}_2(\mathbf{R}) \to P_2(\mathbf{R})$ by $T(p) = p + p'$. You can check that T is a linear transformation. Since $\{\,1, x, x^2\,\}$ is a basis for $\mathrm{P}_2(\mathbf{R})$, $\mathrm{R}(T) = \mathrm{Span}(\{\,T(1), T(x), T(x^2)\,\}) = \mathrm{Span}(\{\,1, x+1, x^2+2x\,\})$. Since it is easy to see that $\{\,1, x+1, x^2+2x\,\}$ is linearly independent, and since $\dim(\mathrm{P}_2(\mathbf{R})) = 3$, it follows that $\mathrm{R}(T) = \mathrm{P}_2(\mathbf{R})$. Hence T is onto and therefore one-to-one as well.

Generating Linear Transformations

Theorem

Let V and W be vector spaces over a field $\mathbf{F}$. Suppose that $\{ v_1, \dots, v_n \}$ is a basis for V. Given vectors $w_1, \dots, w_n$ in W (not necessarily distinct), there is a unique linear transformation $T : V \to W$ such that $T(v_k) = w_k$ for $1 \le k \le n$.

Proof.

If $x \in V$, then there are unique scalars b_k such that $x = \sum_{k=1}^n b_k v_k$. Therefore we can define a function $T : V \to W$ by setting $T(x) = \sum_{k=1}^n b_k w_k$. Of course, $T(v_k) = w_k$.

I claim that T is linear. Suppose $x, y \in V$ and $a \in \mathbf{F}$. Say $x = \sum_{k=1}^n b_k v_k$ and $y = \sum_{k=1}^n c_k v_k$. Then $ax + y = \sum_{k=1}^n (ab_k + c_k) v_k$ and by definition $T(ax + y) = \sum_{k=1}^n (ab_k + c_k) w_k = a \sum_{k=1}^n b_k v_k + \sum_{k=1}^n c_k v_k = aT(x) + T(y)$. This proves the claim.

Proof

Proof Continued.

We still need to see that this uniquely determines T. Suppose that $S : V \to W$ is linear and $S(v_k) = w_k$ for $1 \leq k \leq n$. Then if $x \in \sum_{k=1}^n b_k v_k \in V$,
$S(x) = S\big(\sum_{k=1}^n b_k v_k\big) = \sum_{k=1}^n b_k S(v_k) = \sum_{k=1}^n b_k w_k = T(x)$.
That is, $S = T$ and T is uniquely determined. $\square$

Corollary

Suppose that V and W are vector spaces over $\mathbf{F}$ and that both $T : V \to W$ and $S : V \to W$ are linear. If $\{ v_1, \dots, v_n \}$ is a basis for V, then $S = T$ if and only if $S(v_k) = T(v_k)$ for all $1 \leq k \leq n$.

Break Time Again

Let's take a short break. Any questions?

Order Matters!

Definition

An ordered basis for a finite dimensional vector space V of dimension n is a basis $\{ v_1, v_2, \dots, v_n \}$ considered as an (ordered) n-tuple.

Remark

Both $\{ e_1, e_2, e_3 \}$ and $\{ e_3, e_2, e_1 \}$ are the standard basis for $\mathbf{F}^3$. But they are different as ordered bases. Naturally, we call $\{ e_1, e_2, \dots, e_n \}$ the standard ordered basis for $\mathbf{F}^n$ and $\{ 1, x, \dots, x^n \}$ the standard ordered basis for $\mathrm{P}_n(\mathbf{F})$. Some care is needed for $\mathrm{P}_n(\mathbf{F})$. Unlike the case for $\mathbf{F}^n$, $\{ x^n, x^{n-1}, \dots, x, 1 \}$ is would also be a natural choice—just not the choice we have made.

Coordinate Vectors

Notation

Let $\beta = \{ v_1, \dots, v_n \}$ be an ordered basis for a vector space V. Then for each $x \in V$, there is a unique vector $(a_1, \dots, a_n) \in \mathbf{F}^n$—that is, an ordered n-tuple—such that

$$x = \sum_{k=1}^{n} a_k v_k.$$

We call $(a_1, \dots, a_n)$ the coordinate vector of x relative to β. We use the notation $[x]_\beta = (a_1, \dots, a_n)$ or

$$[x]_\beta = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}.$$

Examples

Example

Let $\beta = \{ 1, x, x^2 \}$ be the standard ordered basis for $\mathrm{P}_2(\mathbf{F})$. Then $[2x^2 - 3x + 7]_\beta = (7, -3, 2)$. (Note the order!) Or, as in the text,

$$[2x^2 - 3x + 7]_\beta = \begin{pmatrix} 7 \\ -3 \\ 2 \end{pmatrix}.$$

Example

Let $\sigma = \{ e_1, e_2, e_3 \}$ be the standard ordered basis for $\mathbf{F}^3$. Then $[(1, 2, 3)]_\sigma = (1, 2, 3)$. This is because $(1, 2, 3) = e_1 + 2e_2 + 3e_3$. In general, if σ is the standard ordered basis for $\mathbf{F}^n$, then $[x]_\sigma = x$ for all $x \in \mathbf{F}^n$. This rather boring observation will nevertheless be useful down the road.

The Matrix of a Linear Transformation

Definition

Suppose that V and W are finite-dimensional vector spaces with ordered bases $\beta = \{ v_1, \dots, v_n \}$ and $\gamma = \{ w_1, \dots, w_m \}$, respectively. (Note that $\dim(V) = n$ and $\dim(W) = m$.) Suppose that $T : V \to W$ is a linear transformation. Then the matrix of T with respect to β and γ is the $m \times n$-matrix $[T]_\beta^\gamma$ whose j^{th}-column is the coordinate vector $[T(v_j)]_\gamma$. When $V = W$ and $\beta = \gamma$, then we usually write $[T]_\beta$ in place of $[T]_\beta^\beta$.

Remark

This is easier to make sense of if we agree—as do the authors of our text—to think $[T(v_j)]_\gamma$ as a column vector. Then

$$[T]_\beta^\gamma = \big[[T(v_1)]_\gamma \; [T(v_2)]_\gamma \; \cdots \; [T(v_n)]_\gamma \big].$$

Take careful note of the shape of $[T]_\beta^\gamma$ as well as the placement of the superscript and subscript!

Example

Define $T : P_2(\mathbf{R}) \to P_3(\mathbf{R})$ by $T(p) = \int_0^x p(t)\,dt$. Let σ_2 and σ_3 be the standard ordered bases for $P_2(\mathbf{R})$ and $P_3(\mathbf{R})$. Find the matrix $[T]_{\sigma_2}^{\sigma_3}$.

Solution

Recall that $\sigma_2 = \{1, x, x^2\}$. *Then* $T(1) = x$. *Therefore* $[T(1)]_{\sigma_3} = (0, 1, 0, 0)$. *We have* $T(x) = \frac{1}{2}x^2$ *while* $T(x^2) = \frac{1}{3}x^3$. *Hence*

$$[T]_{\sigma_2}^{\sigma_3} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}.$$

Standard Matrix

Example

Let $T : \mathbf{R}^3 \to \mathbf{R}^2$ be the linear transformation $T(x, y, z) = (x + y - z, y + 3z)$. Find the matrix of T with respect to the standard ordered bases σ_3 and σ_2.

Solution

Here $T(e_1) = (1, 0)$, $T(e_2) = (1, 1)$, *and* $T(e_3) = (-1, 3)$. *Then* $[T]_{\sigma_3}^{\sigma_2} = \big[[T(e_1)]_{\sigma_2}\ [T(e_2)]_{\sigma_2}\ [T(e_3)]_{\sigma_2}\big]$. *Since* $[x]_{\sigma_2} = x$ *for all* $x \in \mathbf{R}^2$, *the rest is routine:*

$$[T]_{\sigma_3}^{\sigma_2} = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 3 \end{pmatrix}.$$

Enough

1. That is enough for today.