Math 24: Winter 2021
Lecture 10

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Let's Get Started

1. We should be recording.
2. Remember, it is more comfortable for me if you turn on your video so that I feel like I am talking to real people.
3. The preliminary exam is due Saturday by 10pm.
4. I suggest blocking out a three hour window now so that you can work undisturbed. Although officially a take home exam, it is effectively an "in class" exam where you get to pick when you take it.
5. But first, are there any questions from last time?

Review

Theorem

Suppose that V, W, and Z are finite-dimensional vector spaces with ordered bases β, γ, and α, respectively. Suppose that $T : V \to W$ and $S : W \to Z$ are linear. Then

$$[ST]_\beta^\alpha = [S]_\gamma^\alpha [T]_\beta^\gamma.$$

Proposition

Suppose that $A = [u_1 \ \cdots \ u_n]$ is a $m \times n$ matrix with columns $u_1, \dots, u_n$. If B is a $p \times m$-matrix, then BA is the $p \times n$ matrix with columns equal to Bu_j. That is, in my notation,

$$BA = B[u_1 \ \cdots \ u_n] = [Bu_1 \ Bu_2 \ \cdots \ Bu_n].$$

Another Big Pay-Off

Theorem

Suppose that V and W are finite-dimensional vector spaces over $\mathbf{F}$ *and that $T : V \to W$ is a linear transformation. Let β and γ be ordered bases for V and W, respectively. Then for all $v \in V$, we have*

$$[T(v)]_\gamma = [T]_\beta^\gamma [v]_\beta$$

Remark

You would be wise to note carefully the placement of the superscripts and subscripts.

Proof

Proof.

Fix $v \in V$. Let $U : \mathbf{F} \to V$ be the linear map given by $U(a) = av$. Let $\alpha = \{1\}$ be the standard ordered basis for $\mathbf{F}$. Then $TU : \mathbf{F} \to W$ is linear and

$$\begin{aligned}[T(v)]_\gamma &= [TU(1)]_\gamma = [TU]_\alpha^\gamma \\ &= [T]_\beta^\gamma [U]_\alpha^\beta = [T]_\beta^\gamma [U(1)]_\beta \\ &= [T]_\beta^\gamma [v]_\beta.\end{aligned}$$

□

Example

Let $A = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \end{pmatrix}$. Then if $x = (a, b, c) \in \mathbf{F}^3$, we get a vector $Ax \in \mathbf{F}^2$ by $\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a-2c \\ b+c \end{pmatrix}$. Since the rules of matrix algebra imply that $A(kx + y) = kAx + Ay$, it is easy to see that this map is linear from $\mathbf{F}^3$ to $\mathbf{F}^2$. This is an easy way to produce linear maps, and we will see that all linear maps between finite-dimensional vector spaces are "essentially of this form".

Left-Multiplication

Definition

Let $A \in M_{m\times n}(\mathbf{F})$. Then we obtain a map $L_A : \mathbf{F}^n \to \mathbf{F}^m$ defined by $L_A(x) = Ax$ where we view $x \in \mathbf{F}^n$ as a $n \times 1$-matrix. We call L_A the left-multiplication transformation determined by A.

Theorem

Let $A \in M_{m\times n}(\mathbf{F})$. Then $L_A : \mathbf{F}^n \to \mathbf{F}^m$ is linear. Moreover, if σ_n and σ_m are the standard ordered bases for $\mathbf{F}^n$ and $\mathbf{F}^m$, respectively, then

1. $[L_A]_{\sigma_n}^{\sigma_m} = A$,
2. $L_A = L_B$ *if and only if* $A = B$,
3. $L_{aA+B} = aL_A + L_B$,
4. *if* $T : \mathbf{F}^n \to \mathbf{F}^m$ *is linear, then there is a unique* $C \in M_{m\times n}(\mathbf{F})$ *such that* $T = L_C$ *and* $C = [T]_{\sigma_n}^{\sigma_m}$,
5. *If* $E \in M_{n\times p}(\mathbf{F})$, *then* $L_{AE} = L_A L_E$, *and*
6. *if* $m = n$, *then* $L_{I_n} = I_{F_n}$.

Proof

Proof.

Just as in the previous example, the rules of matrix algebra tell us that L_A is a linear transformation from $\mathbf{F}^n$ to $\mathbf{F}^m$.

(1) The j^{th}-column of $[L_A]_{\sigma_n}^{\sigma_m}$ is $[L_A e_j]_{\sigma_m} = [Ae_j]_{\sigma_m} = Ae_j$, which is the j^{th}-column of A.

(2) If $L_A = L_B$, then $A = [L_A]_{\sigma_n}^{\sigma_m} = [L_B]_{\sigma_n}^{\sigma_m} = B$. The converse is immediate.

(3) By item (1), $[L_{aA+B}]_{\sigma_n}^{\sigma_m} = aA + B$ and $[aL_A + L_B]_{\sigma_n}^{\sigma_m} = a[L_A]_{\sigma_n}^{\sigma_m} + [L_B]_{\sigma_n}^{\sigma_m} = aA + B$. Now the result follows from item (2).

(4) Let $C = [T]_{\sigma_n}^{\sigma_m}$. Then for all $x \in \mathbf{F}^n$, $Tx = [Tx]_{\sigma_m} = [T]_{\sigma_n}^{\sigma_m}[x]_{\sigma_n} = Cx = L_C x$. Hence $T = L_C$. Uniqueness follows from item (2).

(5) Let σ_p be the standard ordered basis for $\mathbf{R}^p$. Then on the one hand, $[L_{AE}]_{\sigma_p}^{\sigma_m} = AE$ by item (1). On the other hand, $[L_A L_E]_{\sigma_p}^{\sigma_m} = [L_A]_{\sigma_n}^{\sigma_m}[L_E]_{\sigma_p}^{\sigma_n} = AE$. Now the result follows from item (2).

(6) This is straightforward. □

A Missing Proof

Proposition

Suppose that A, B, and C are matrices such that $(AB)C$ is defined. Then $A(BC)$ is defined and $(AB)C = A(BC)$.

Proof.

It is an exercise to check that $A(BC)$ defined if and only if $(AB)C$ is. Also, it is clear that $(L_A L_B)L_C = L_A(L_B L_C)$. Now pass to matrices using the previous result. □

Break Time

Time for a break and some questions.

Invertible Maps

Definition

A linear map $T : V \to W$ is said to be invertible if there is a function $U : W \to V$ such that $UT = I_V$ and $TU = I_W$. Then we call U an inverse for T.

Proposition

Suppose that $T : V \to W$ is linear. Then T is invertible if and only if T is one-to-one and onto. Furthermore, the inverse U is unique and linear. We usually denote the inverse by T^{-1}.

Proof

Proof.

Suppose that T is one-to-one and onto. Then we can define $U : W \to V$ by $U(w) = v$ if and only if $T(v) = w$. Then $UT(v) = v$ and $TU(w) = w$. Hence T is invertible.

On the other hand, suppose T is invertible with inverse U. If $T(x) = T(y)$, then $x = UT(x) = UT(y) = y$ and T is one-to-one. If $w \in W$, then $T(U(w)) = w$ and T is onto. This proves the first assertion.

Suppose that S is also an inverse to T. Then $S = SI_W = S(TU) = (ST)U = I_V U = U$. Thus the inverse unique.

Suppose $w, w' \in W$. Then $w = T(v)$ and $w' = T(v')$. Then $T(av + v') = aw + w'$. Therefore $U(aw + w') = U(aT(v) + T(v')) = UT(av + v') = av + v' = aU(w) + U(w')$ and U is linear. □

Isomorphism

Definition

If V and W are vector spaces over $\mathbf{F}$, then we say that V and W are isomorphic if there is an invertible linear transformation $T : V \to W$. In that case, we call T an isomorphism of V onto W.

Remark

Note that if $T : V \to W$ is an isomorphism of V onto W, then $T^{-1} : W \to V$ is an isomorphism of W onto V. Hence the situation is symmetric.

Isomorphism and Dimension

Proposition

Suppose that V and W are isomorphic. Then V is finite dimensional if and only if W is. In that case, $\dim(V) = \dim(W)$*.*

Proof.

Let $T : V \to W$ be an isomorphism. Suppose that β is finite basis for V, then $T(\beta)$ generates $R(T) = W$. Hence W is finite dimensional. Since we can reverse the roles of V and W, this proves the first statement. Now if V and W are finite dimensional, we have $\dim(V) = \text{nullity}(T) + \text{rank}(T)$. Since T is one-to-one, $\text{nullity}(T) = 0$ and $\text{rank}(T) = \dim(V)$. Since T is onto, $\text{rank}(T) = \dim(R(T)) = \dim(W)$. Hence $\dim(V) = \dim(W)$. □

Break Time

Time for a break and questions.

Invertible Matrices

Definition

A matrix $A \in M_{n\times n}(\mathbf{F})$ is invertible if there is a matrix $B \in M_{n\times n}(\mathbf{F})$ such that $AB = I_n = BA$.

Remark

As with linear maps, if B exists, it is unique. If C were another such matrix, then $C = CI_n = C(AB) = (CA)B = I_nB = B$. Hence we call B the inverse of A and foolishly write A^{-1} in place of B.

Example

Example

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and if $ad - bc \neq 0$, then you can verify (by multiplying matrices) that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

As we will discover in the next chapter, finding when inverses exist and computing them is something of a chore.

Playing Together

Theorem

Suppose that V and W are finite-dimensional vector spaces over with ordered bases β and γ, respectively. Then a linear map $T : V \to W$ is invertible if and only if $[T]_\beta^\gamma$ is invertible. In this case, $[T^{-1}]_\gamma^\beta = \big([T]_\beta^\gamma\big)^{-1}$.

Proof.

If T is invertible, then we have already seen that $\dim(V) = \dim(W)$. Hence $[T]_\beta^\gamma$ is a square matrix. Let $\dim(V) = n$. Then

$$I_n = [I_V]_\beta = [T^{-1}T]_\beta^\beta = [T^{-1}]_\gamma^\beta [T]_\beta^\gamma$$

Similarly, $[I_n] = [I_W]_\gamma = [T]_\beta^\gamma [T^{-1}]_\gamma^\beta$. It follows that $[T]_\beta^\gamma$ is invertible with inverse $[T^{-1}]_\gamma^\beta$.

Proof

Proof.

Now suppose that $[T]_\beta^\gamma$ is invertible with inverse $B = (B_{ij})$ (so that $B[T]_\beta^\gamma = [T]_\beta^\gamma B = I_n$). Let $\beta = \{ v_1, \dots, v_n \}$ and $\gamma = \{ w_1, \dots, w_n \}$. Let S be the unique linear transformation $S : W \to V$ such that $S(w_j) = \sum_{k=1}^n B_{kj} v_k$. Then $[S]_\gamma^\beta = B$. Then

$$[ST]_\beta = [S]_\gamma^\beta [T]_\beta^\gamma = B[T]_\beta^\gamma = I_n.$$

Therefore $ST = I_V$. Similarly, $TS = I_W$ and S is an inverse for T. That is, T is invertible as claimed. □

Corollary

If $A \in M_{n\times n}(\mathbf{F})$, then A is invertible if and only if L_A is and $L_A^{-1} = L_{A^{-1}}$.

Once You've Seen One, . . .

Theorem

Suppose that V and W are finite-dimensional vector spaces over the same field $\mathbf{F}$. Then V is isomorphic to W if and only if $\dim(V) = \dim(W)$.

Proof.

We have already see that if V and W are isomorphic, then $\dim(V) = \dim(W)$. So suppose that $\dim(V) = \dim(W)$. Let $\beta = \{ v_1, \ldots, v_n \}$ be a basis for V and $\gamma = \{ w_1, \ldots, w_n \}$ be a basis for W. Let $T : V \to W$ be the unique linear transformation such that $T(v_k) = w_k$ for $1 \leq k \leq n$. Then $\mathrm{R}(T) = \mathrm{Span}(T(\beta)) = \mathrm{Span}(\gamma) = W$. Therefore T is onto. Since $\dim(V) = \dim(W)$, T is also one-to-one. Then T is invertible and hence an isomorphism. □

Corollary

Let V be a vector space over $\mathbf{F}$. Then $\dim(V) = n$ if and only if V is isomorphic to $\mathbf{F}^n$.

Remark

We have the tools to say a bit more. If $\beta = \{ v_1, \dots, v_n \}$ is a basis for V, then you showed on homework that $\varphi_\beta(x) := [x]_\beta$ is an onto linear transformation of V onto $\mathbf{F}^n$. Since $\dim(V) = n = \dim(\mathbf{F}^n)$, φ_β is an isomorphism called the standard representation of V with respect to β.

Remark

Suppose that V and W are finite-dimensional vector spaces over $\mathbf{F}$ with $\dim(V) = n$ and $\dim(W) = m$. Then we showed earlier that $T \mapsto [T]_\beta^\gamma$ is a one-to-one and onto linear transformation of $\Phi : \mathcal{L}(V, W) \to M_{m\times n}(\mathbf{F})$. Hence $\mathcal{L}(V, W)$ and $M_{m\times n}(\mathbf{F})$ are isomorphic and $\dim(\mathcal{L}(V, W)) = \dim(M_{m\times n}(\mathbf{F})) = mn$.

Enough

1. That is enough for today.