Math 24: Winter 2021
Lecture 11

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Let's Get Started

1. We should be recording.
2. Remember, it is more comfortable for me if you turn on your video so that I feel like I am talking to real people.
3. The preliminary exam has been returned via `gradescope`. I have sent and email with solutions and comments.
4. All solutions should be formulated clearly using full sentences. In particular, your solutions should not look like a first draft! I hope we will do better on the midterm.
5. Do not submit regrade requests. After studying the solutions, you can arrange a zoom meeting to discuss your exam.
6. But first, are there any questions from last time?

Reveiw

Definition

If V and W are vector spaces over $\mathbf{F}$, then we say that V and W are isomorphic if there is an invertible linear transformation $T : V \to W$. In that case, we call T an isomorphism of V onto W.

Definition

A matrix $A \in M_{n\times n}(\mathbf{F})$ is invertible if there is a matrix $B \in M_{n\times n}(\mathbf{F})$ such that $AB = I_n = BA$. Then B, if it exists, is unique. We call B the inverse of A and write A^{-1} for B.

Review

Theorem

Suppose that V and W are finite-dimensional vector spaces over with ordered bases β and γ, respectively. linear map $T : V \to W$ is invertible if and only if $[T]_\beta^\gamma$ is invertible. In this case, $[T^{-1}]_\gamma^\beta = \big([T]_\beta^\gamma\big)^{-1}$.

Theorem

Suppose that V and W are finite-dimensional vector spaces over the same field $\mathbf{F}$. Then V is isomorphic to W if and only if $\dim(V) = \dim(W)$.

Our Favorite

Corollary

Let V be a vector space over $\mathbf{F}$. Then $\dim(V) = n$ if and only if V is isomorphic to $\mathbf{F}^n$.

Remark

We have the tools to say a bit more. If $\beta = \{ v_1, \dots, v_n \}$ is an ordered basis for V, then you showed on homework that $\varphi_\beta(x) := [x]_\beta$ is an onto linear transformation of V onto $\mathbf{F}^n$. Since $\dim(V) = n = \dim(\mathbf{F}^n)$, φ_β is an isomorphism called the standard representation of V with respect to β.

A Nice Diagram

Remark (A Useful Picture)

Suppose V and W are finite dimensional with ordered bases $\beta = \{ v_1, \dots, v_n \}$ and $\gamma = \{ w_1, \dots, w_m \}$, respectively. Suppose that $T : V \to W$ is linear. Since $[T(v)]_\gamma = [T]_\beta^\gamma [v]_\beta$, we have the following nice picture:

$$\begin{array}{ccc} V & \xrightarrow{\quad T \quad} & W \\ {\scriptstyle \varphi_\beta}\downarrow & & \downarrow{\scriptstyle \varphi_\gamma} \\ \mathbf{F}^n & \xrightarrow[\quad L_{[T]_\beta^\gamma} \quad]{} & \mathbf{F}^m. \end{array}$$

where the vertical arrows are the standard representation isomorphisms.

Unfinished Business

Theorem

Suppose $A, B \in M_{n\times n}(\mathbf{F})$. If $AB = I_n$, then both A and B are invertible with $B = A^{-1}$ and $A = B^{-1}$.

Proof.

Let $L_A : \mathbf{F}^n \to \mathbf{F}^n$ the left-multiplication transformation. Fix $x \in \mathbf{F}^n$. Then $Bx \in \mathbf{F}^n$ and $L_A(Bx) = A(Bx) = (AB)x = I_n x = x$. Therefore L_A is onto. Since L_A maps $\mathbf{F}^n$ to itself, L_A must also be one-to-one. Therefore L_A is invertible which implies A is invertible. Then $A^{-1} = A^{-1}I_n = A^{-1}(AB) = B$. Then B is invertible (since A^{-1} is) and $B^{-1} = (A^{-1})^{-1} = A$. □

Remark

Suppose that V and W are finite-dimensional vector spaces over $\mathbf{F}$ with $\dim(V) = n$ and $\dim(W) = m$. Then we showed earlier that $T \mapsto [T]_\beta^\gamma$ is a one-to-one and onto linear transformation of $\Phi : \mathcal{L}(V, W) \to M_{m\times n}(\mathbf{F})$. Hence $\mathcal{L}(V, W)$ and $M_{m\times n}(\mathbf{F})$ are isomorphic and $\dim(\mathcal{L}(V, W)) = \dim(M_{m\times n}(\mathbf{F})) = mn$.

Break Time

This finishes §2.4 in the text. We'll finish §2.5 next and skip §2.6 and §2.7 and move onto §3.1.

Now let's take a break and see if there are any questions.

The Matrix of the Identity Transformation

Remark

Let $\beta = \{ v_1, \dots, v_n \}$ be an ordered basis for V. Then it is straightforward to see that $[I_V]_\beta = I_n$. But if γ is another ordered basis, a little thought should reveal that $[I_V]_\beta^\gamma$ is unlikely to be the identity matrix or even easy to compute—since $[I_V]_\beta^\gamma = \big[[v_1]_\gamma \ \cdots \ [v_n]_\gamma \big]$, we would have to compute $[v_k]_\gamma$ for $k = 1, 2, \dots, n$. But it just might turn out to be worth the effort.

Change of Basis Matrices

Proposition

Suppose that β and γ are both ordered bases for a vector space V. Then for all $v \in V$,

$$[v]_\gamma = [I_V]_\beta^\gamma [v]_\beta. \tag{‡}$$

Hence we call $Q = [I_V]_\beta^\gamma$ the change of coordinates matrix from β-coordinates to γ-coordinates. Furthermore, $[I_V]_\beta^\gamma$ is invertible with $\big([I_V]_\beta^\gamma\big)^{-1} = [I_V]_\gamma^\beta$ which is the change of coordinate matrix from γ-coordinates to β-coordinates.

Proof

Proof.

We have

$$[v]_\gamma = [I_V(v)]_\gamma = [I_V]_\beta^\gamma [v]_\beta.$$

This establishes (‡). Of course $[I_V]_\beta^\gamma$ is invertible because I_V is and $\left([I_V]_\beta^\gamma\right)^{-1} = [I_V^{-1}]_\gamma^\beta = [I_V]_\gamma^\beta.$ □

Example

Recall that if $x \in \mathbf{F}^n$ and σ is the standard ordered basis for $\mathbf{F}^n$, then $[x]_\sigma = x$. This leads to another useful observation. Let $\beta = \{ v_1, \dots, v_n \}$ another ordered basis for $\mathbf{F}^n$. Then $[I]_\beta^\sigma$ is easy to compute (where I've written I in place of $I_{\mathbf{F}^n}$ for obvious reasons)!

$$\begin{aligned} [I_{\mathbf{F}^n}]_\beta^\sigma &= \big[[I(v_1)]_\sigma \ \cdots \ [I(v_n)]_\sigma \big] \\ &= \big[[v_1]_\sigma \ \cdots \ [v_n]_\sigma \big] = \big[v_1 \ \cdots \ v_n \big] \end{aligned}$$

Example

Let $\beta = \{ (2,1), (1,1) \}$ and $\gamma = \{ (3,-2), (-1,1) \}$ be ordered bases for $\mathbf{R}^2$. Find the change of coordinate matrix $[I]_\beta^\gamma$.

Solution

It is not too bad to find $\left[\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right]_\gamma$ *and* $\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right]_\gamma$. *For example, you just have to solve* $a(3,-2) + b(-1,1) = (2,1)$ *and then* $\left[\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right]_\gamma = \begin{pmatrix} a \\ b \end{pmatrix}$. *But we can also note that if* σ *is the standard ordered basis then*

$$\begin{aligned} [I]_\beta^\gamma &= [I]_\sigma^\gamma [I]_\beta^\sigma = \left([I]_\gamma^\sigma\right)^{-1} [I]_\beta^\sigma \\ &= \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 7 & 5 \end{pmatrix}. \end{aligned}$$

Break Time

Time for a short break and some questions.

Change of Basis

Remark

Much of our attention going forward will be on linear maps $T : V \to V$ from a vector space to itself. We call such maps linear operators. A key question for us will be to see what sort of matrices we get for a linear operator for different choices of basis for V.

Theorem (Change of Basis)

Suppose that $T : V \to V$ is a linear operator on a finite-dimensional vector space. Let β and γ both be ordered basis for V. Let $Q = [I_V]_\gamma^\beta$ be the change of coordinates matrix from γ-coordinates to β-coordinates. Then

$$[T]_\gamma = Q^{-1}[T]_\beta Q.$$

Proof

Proof.

We have

$$\begin{aligned} [T]_\gamma &= [I_V T]_\gamma^\gamma = [I_V]_\beta^\gamma [T]_\gamma^\beta \\ &= [I_V]_\beta^\gamma [T I_V]_\gamma^\beta = [I_V]_\beta^\gamma [T]_\beta^\beta [I_V]_\gamma^\beta \\ &= \big([I_V]_\gamma^\beta\big)^{-1} [T]_\beta^\beta [I_V]_\gamma^\beta = Q^{-1}[T]_\beta Q. \end{aligned}$$

□

Remark

I have introduced the notation $Q = [I_V]_\gamma^\beta$ only because the text does. I will usually write $[I_V]_\gamma^\beta$ as I think the meaning is clearer.

Example

Example

Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the left multiplication operator L_A for $A = \begin{pmatrix} 7 & -10 \\ 5 & -8 \end{pmatrix}$. That is, $T(x,y) = \begin{pmatrix} 7x-10y \\ 5x-8y \end{pmatrix}$. Let β be the ordered basis $\{(2,1),(1,1)\}$. Find $[T]_\beta$.

Solution

Let σ be the standard basis for $\mathbf{R}^2$. Then

$$\begin{aligned} [T]_\beta &= [I]_\sigma^\beta [T]_\sigma [I]_\beta^\sigma = \left([I]_\beta^\sigma\right)^{-1} A [I]_\beta^\sigma \\ &= \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 7 & -10 \\ 5 & -8 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 4 & -3 \\ 2 & -3 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix} \end{aligned}$$

Remark

Naturally, the properties of the operator T are much easier to understand if we use β-coordinates. One of our goals down the road will be to discover how to find β!!

Remark

The map $P : \mathbf{R}^2 \to \mathbf{R}^2$ given by $P(x, y) = (x, 0)$ is the projection of $\mathbf{R}^2$ onto the subspace $W_1 = \{ (x, 0) : x \in \mathbf{R} \}$ along the subspace $W_2 = \{ (0, y) : y \in \mathbf{R} \}$. Of course if σ is the standard ordered basis in $\mathbf{R}^2$, $[P]_\sigma = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. More generally, we can consider the ordered basis $\beta = \{ u_1, u_2 \}$. Then we can let $W_1 = \text{Span}(\{u_1\})$ and $W_2 = \text{Span}(\{u_2\})$. Then $\mathbf{R}^2 = W_1 \oplus W_2$. (You should check this.) Then we can consider the projection P of $\mathbf{R}^2$ onto W_1 along W_2. Thus if $v = au_1 + bu_2$, then $P(v) = au_1$. Just as above, we have $[P]_\beta = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. But for practical purposes, we'd much rather have a formula for $[P]_\sigma$! This is what the Change of Basis Theorem is for (among other things)!

Working Out an Example

Example

To make things concrete, let $\beta = \{ (1,3), (1,1) \}$ and consider the project P of $\mathbf{R}^2$ onto the span of $(1,3)$ along the span of $(1,1)$. (This can also be described as the projection onto the line $y = 3x$ along the line $y = x$.) But

$$\begin{aligned} [P]_\sigma &= [I]^\sigma_\beta [P]_\beta [I]^\beta_\sigma \\ &= \begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 1 & 0 \\ 3 & 0 \end{pmatrix} \left(-\tfrac{1}{2}\right) \begin{pmatrix} 1 & -1 \\ -3 & 1 \end{pmatrix} \\ &= \left(-\tfrac{1}{2}\right) \begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix}. \end{aligned}$$

Hence $P(x,y) = \begin{pmatrix} -\frac{1}{2}(x-y) \\ -\frac{3}{2}(x-y) \end{pmatrix}$.

Enough

1. That is enough for today.