Math 24: Winter 2021
Lecture 12

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Let's Get Started

1. We should be recording.
2. Remember, it is more comfortable for me if you turn on your video so that I feel like I am talking to real people.
3. Homework #4 is due today.
4. But first, are there any questions from last time?

Change of Basis

Remark

Much of our attention going forward will be on linear maps $T : V \to V$ from a vector space to itself. We call such maps linear operators. A key question for us will be to see what sort of matrices we get for a linear operator for different choices of basis for V.

Theorem (Change of Basis)

Suppose that $T : V \to V$ is a linear operator on a finite-dimensional vector space. Let β and γ both be ordered basis for V. Let $Q = [I_V]_\gamma^\beta$ be the change of coordinates matrix from γ-coordinates to β-coordinates. Then

$$[T]_\gamma = \big([I_V]_\beta^\gamma\big)^{-1}[T]_\beta[I_V]_\gamma^\beta.$$

Motivation

1. I should be clear now—especially in view of the Change of Basis Theorem—that the matrices $[I_V]_\beta^\gamma$ and their inverses will play a big role going forward.
2. Computing $[I_V]_\beta^\gamma$ will involve systems of linear equations.
3. We will also want to determine when a matrix is invertible and how to compute the inverse when it exists.

Systems

Formally, a system of m linear equations in n unknowns $x_1, \dots, x_n$ over a field $\mathbf{F}$ can be written as follows:

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad & \quad \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

where the a_{ij} and b_k are scalars in $\mathbf{F}$. A solution is any vector $(x_1, \dots, x_n) \in \mathbf{F}^n$ that satisfies each equation simultaneously. Later in this chapter, we will write this system more compactly as

$$Ax = b$$

where $A = (a_{ij})$ is the $m \times n$ matrix with $(i, j)^{\text{th}}$-entry a_{ij}, $x = (x_1, \dots, x_n)$ and $b = (b_1, \dots, b_m)$.

Elementary Operations

Recall that we noted earlier that we could obtain systems of equations with the same sets of solutions if we restricted ourselves to three basic—or elementary—operations:

1. Interchanging two equations.
2. Multiplying an equation by a nonzero scalar.
3. Adding a multiple of one equation to a different equation.

Since we are going to focus on matrices for a bit, this serves to motivate the following definition.

Elementary Row and Column Operations

Definition

Let A be a $m \times n$-matrix. An elementary row [column] operation on A is one of the following.

- Type 1: Interchanging two rows [columns].
- Type 2: Multiplying a row [column] by a nonzero scalar.
- Type 3: Adding a multiple of one row [column] to another row [column].

Example

Working out examples is easy (Document Camera).

Reversible

Lemma

If we can obtain a matrix Q from a matrix P by an elementary row [column] operation, then we can obtain P from Q by an elementary row [column] operation of the same type. Thus elementary operations are always reversible.

Proof.

I will leave this as an exercise. □

Elemenatary Matrices

Definition

An elementary matrix is a $n \times n$-matrix obtained from the identity matrix I_n be a single elementary row or column operation. An elementary matrix is said to be of type 1, type 2, or type 3 depending on the type of elementary operation performed on I_n.

Examples

Back to the Document Camera.

Remark

It is worth noting that every elementary matrix can be obtained two ways—either by an elementary row operation or and elementary column operation.

What's so Cool About Elementary Matrices?

Theorem

Suppose that A is a $m \times n$-matrix. If B is the $m \times n$-matrix obtained from A via an elementary row operation, then $B = EA$ where E is the $m \times m$-elementary matrix obtained from I_m via the same elementary row operation. Similarly, if C is obtained from A via an elementary column operation, then $C = AD$ where D is an elementary $n \times n$-matrix obtained from I_n via the same elementary column operation.

Proof.

The proof is a bit tedious and not particularly enlightening, so we will omit it. □

Elementary Inverses

Proposition

Every elementary matrix is invertible and its inverse is an elementary matrix of the same type.

Proof.

Let E be an elementary $n \times n$-matrix. If E was obtained from I_n by an elementary row operation, then we can obtain I_n from E by performing an elementary row operation of the same type. By our theorem, there is a $n \times n$-elementary matrix D of the same type as E such that $DE = I_n$. But we have proved this means E is invertible with inverse $E^{-1} = D$. □

Break Time

Time for a break and some questions.

Rank of a Matrix

Definition

If $A \in M_{m \times n}(\mathbf{F})$, then its rank—written $\operatorname{rank}(A)$—is the rank of the associated left-multiplication operator $L_A : \mathbf{F}^n \to \mathbf{F}^m$.

Remark

Thus $\operatorname{rank}(A) = \dim(R(L_A))$. Therefore $\operatorname{rank}(A) \leq m$ and equals m exactly when L_A is onto.

Rank and Invertibility

Proposition

Let A be a $n \times n$ matrix. Then A is invertible if and only if $\operatorname{rank}(A) = n$.

Proof.

If $\operatorname{rank}(A) = n$, then $L_A : \mathbf{F}^n \to \mathbf{F}^n$ is onto. Then L_A is also one-to-one and L_A is invertible. Therefore A is invertible.

But if A is invertible, then so is L_A. Hence L_A is onto and $\operatorname{rank}(A) = n$. □

Theorem

Suppose that $T : V \to W$ is a linear transformation. Suppose that V and W are finite dimensional with ordered bases β and γ, respectively. Then

$$\operatorname{rank}(T) = \operatorname{rank}\big([T]_\beta^\gamma\big).$$

Proof.

We need to prove that $\dim(\mathsf{R}(T)) = \dim(\mathsf{R}(L_{[T]_\beta^\gamma}))$. Recall our pretty picture from the last lecture:

$$\begin{array}{ccc} V & \xrightarrow{\quad T \quad} & W \\ {\scriptstyle \varphi_\beta}\downarrow & & \downarrow{\scriptstyle \varphi_\gamma} \\ \mathbf{F}^n & \xrightarrow[L_{[T]_\beta^\gamma}]{} & \mathbf{F}^m. \end{array}$$

where the vertical arrows are the standard representation isomorphisms. This implies that $\varphi_\gamma\big(\mathsf{R}(T)\big) = \mathsf{R}(L_{[T]_\beta^\gamma})$.

Proof

Proof Continued.

Let $\alpha = \{ w_1, \dots, w_k \}$ be a basis for $R(T)$. It is not hard to see that $\varphi_\gamma(\alpha) = \{ \varphi_\gamma(w_1), \dots, \varphi_\gamma(w_k) \}$ spans $R(L_{[T]_\beta^\gamma})$. So it suffices to see that $\{ \varphi_\gamma(w_1), \dots, \varphi_\gamma(w_k) \}$ is linearly independent. So, suppose that there are scalars a_j such that

$$0_{\mathbf{F}^m} = \sum_{j=1}^{k} a_j \varphi_\gamma(w_j) = \varphi_\gamma\Big(\sum_{j=1}^{k} a_j w_j\Big).$$

Since φ_γ is an isomorphism (and hence one-to-one), this implies

$$\sum_{j=1}^{k} a_j w_j = 0_W.$$

Since α is a basis, this implies each $a_j = 0$. This is what we needed to prove. ▸ return □

Rank Preserving

Theorem

Let $A \in M_{m\times n}(\mathbf{F})$ and let P and Q be invertible $m \times m$- and $n \times n$-matrices, respectively. Then

$$\mathrm{rank}(A) = \mathrm{rank}(PA) = \mathrm{rank}(AQ) = \mathrm{rank}(PAQ).$$

Proof.

Since L_Q is onto,

$$\mathrm{R}(L_{AQ}) = \mathrm{R}(L_A L_Q) = L_A\big(L_Q(\mathbf{F}^n)\big) = L_A(\mathbf{F}^n) = \mathrm{R}(L_A).$$

Hence $\mathrm{rank}(AQ) = \mathrm{rank}(A)$.

Since L_P is an isomorphism,
$\dim\big(\mathrm{R}(L_A)\big) = \dim\big(L_P\big(\mathrm{R}(L_A)\big)\big) = \dim(\mathrm{R}(L_{PA}))$. (This is proved just as earlier.) Hence $\mathrm{rank}(A) = \mathrm{rank}(PA)$. But by the above, $\mathrm{rank}(PA) = \mathrm{rank}(PAQ)$. □

Elementary Ranks

Corollary

Elementary row and column operations preserve the rank of a matrix.

Proof.

Elementary row and column operations amount to pre- and post-multiplying by and elementary—and hence invertible—matrix. But if E and D are elementary matrices, then $\operatorname{rank}(A) = \operatorname{rank}(EA)$ and $\operatorname{rank}(A) = \operatorname{rank}(AD)$. □

The Columns of A

Proposition

Let A be a $m \times n$ matrix with columns $v_1, \dots, v_n$. That is, $A = [v_1 \ \cdots \ v_n]$. Then $\mathbf{R}(L_A) = \operatorname{Span}\big(\{ v_1, \dots, v_n \}\big)$

Proof.

We have $\mathrm{R}(L_A) = \{ Ax : x \in \mathbf{F}^n \}$. But if $x \in \mathbf{F}^n$, then $x = (x_1, \dots, x_n) = \sum_{k=1}^n x_k e_k$ where $\{ e_1, \dots, e_n \}$ is the standard basis. Then $Ax = A\big(\sum_{k=1}^n x_k e_k\big) = \sum_{k=1}^n x_k A e_k = \sum_{k=1}^n x_k v_k \in \operatorname{Span}\big(\{ v_1, \dots, v_n \}\big)$. This shows $\mathrm{R}(L_A) \subset \operatorname{Span}\big(\{ v_1, \dots, v_n \}\big)$ and the other containment is also easy to see by reversing the above. $\square$

A Basis for the Range

Corollary

The rank of a $m \times n$*-matrix* $A = [v_1 \ \cdots \ v_n]$ *is the maximum number of linearly independent columns in* A*.*

Proof.

By the previous result, the columns of A, $\{ v_1, \dots, v_n \}$ generate $\mathrm{R}(L_A)$. Hence we know that some subset if a basis for $\mathrm{R}(L_A)$. Thus $\mathrm{rank}(A) = \dim(\mathrm{R}(L_A))$ is the maximum number of linearly independent vectors in $\{ v_1, \dots, v_n \}$. □

Enough

1. That is enough for today.