Math 24: Winter 2021
Lecture 16

Dana P. Williams

Dartmouth College

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Let's Get Started

1. We should be recording.
2. Remember, it is more comfortable for me if you turn on your video so that I feel like I am talking to real people.
3. Our Midterm will be available Thursday after office hours and must be turned in by Saturday, February 20th, at 10pm. No exceptions. More details next week, but it will cover through Monday's lecture—so most likely through §4.3 in the text.
4. But first, are there any questions from last time?

Determinants of Order 2

Definition

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a 2×2-matrix. Then the determinant of A is the scalar

$$\det(A) = ad - bc.$$

The notation $|A|$ is also used.

Remark

Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then $\det(A) = 4 - 6 = -2$, while $\det(B) = 1$. But $A + B = \begin{pmatrix} 2 & 2 \\ 3 & 5 \end{pmatrix}$ and $\det(A + B) = 10 - 6 = 4$. So in this case, $4 = \det(A + B) \neq \det(A) + \det(B) = -2 + 1 = -1$. So even though most things in Math 24 are linear, the determinant is not one of them. But we are not done investigating the determinant yet!

Multi-linear

Remark

The determinant is "more linear" that it first appears. If $u, v \in \mathbf{F}^2$, then we can form a matrix $\binom{u}{v} = \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix}$ with rows u and v. Thus $\det\binom{u}{v} = u_1 v_2 - u_2 v_1$. Then we can define functions T_v and T_u from $\mathbf{F}^2$ to $\mathbf{F}$ by $T_v(x) = \det\binom{x}{v}$ and $T^u(x) = \det\binom{u}{x}$

Proposition

The function $\det : M_{2\times 2}(\mathbf{F}) \to \mathbf{F}$ *is a linear function of each of its rows when the other row is held fixed. That is, for any* $u, v \in \mathbf{F}^2$, *the maps* T_v *and* T^u *defined above are linear maps from* $\mathbf{F}^2$ *to* $\mathbf{F}$.

Remark

This just means that if $a \in \mathbf{F}$ and $x, y, u, v \in \mathbf{F}^2$, then
$\det\binom{ax+y}{v} = a\det\binom{x}{v} + \det\binom{y}{v}$ and
$\det\binom{u}{ax+y} = a\det\binom{u}{x} + \det\binom{u}{y}$.

Proof

Proof.

Let $x = (x_1, x_2)$, $y = (y_1, y_2)$, and $v = (v_1, v_2)$. Then

$$\begin{aligned} T_v(ax + y) &= \det\begin{pmatrix} ax+y \\ v \end{pmatrix} = \det\begin{pmatrix} ax_1+y_1 & ax_2+y_2 \\ v_1 & v_2 \end{pmatrix} \\ &= (ax_1 + y_1)v_2 - v_1(ax_2 + y_2) \\ &= a(x_1v_2 - x_2v_1) + y_1v_2 - y_2v_1 \\ &= a\det\begin{pmatrix} x_1 & x_2 \\ v_1 & v_2 \end{pmatrix} + \det\begin{pmatrix} y_1 & y_2 \\ v_1 & v_2 \end{pmatrix} \\ &= a\det\begin{pmatrix} x \\ v \end{pmatrix} + \det\begin{pmatrix} y \\ v \end{pmatrix} = aT_v(x) + T_v(y). \end{aligned}$$

The proof is similar for T_u. □

Invertibility

Theorem

If $A \in M_{2\times 2}(\mathbf{F})$, then A is invertible if and only if $\det(A) \neq 0$. *If* $A = \det\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ *is invertible, then*

$$A^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Proof.

We already have checked that if $\det(A) \neq 0$, then A is invertible with the inverse given given by the above formula. Conversely, suppose that $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and that $\det(A) = ad - bc = 0$. If $a = b = 0$, then $\operatorname{rank}(A) \leq 1$ and A is not invertible. Otherwise, $(-b, a) \neq 0_{\mathbf{F}^2}$ and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} -b \\ a \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Then L_A has nontrivial kernel and L_A is not one-to-one. Hence A is not invertible. □

Remark on the Text

Remark

If $u, v \in \mathbf{R}^2$, and we view them as vectors in the plane as we did in Math 8, or from Physics class, or just from back in the day, then they determine a parallelogram. In our text, the authors give a rather indirect proof that if $A(u, v)$ is the area of the parallelogram determined by u and v, then

$$A(u, v) = \left| \det \begin{pmatrix} u \\ v \end{pmatrix} \right|.$$

We will accept this as a nice way of visualizing the value of the determinant, but you are not responsible for the treatment in the text.

Break Time

Time for a break and some questions.

Determinants: The General Case

Notation

If $A = (A_{ij})$ is a $n \times n$-matrix, then we let $\widetilde{A}_{ij}$ be the $(n-1) \times (n-1)$-matrix obtained from A be deleting the i^{th}-row and j^{th}-column.

Example

If $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$, then $\widetilde{A}_{23} = \begin{pmatrix} 1 & 2 \\ 7 & 8 \end{pmatrix}$ and $\widetilde{A}_{12} = \begin{pmatrix} 4 & 6 \\ 7 & 9 \end{pmatrix}$. Then $\widetilde{(\widetilde{A}_{12})}_{11} = (9)$.

The Determinant for $n \times n$-Matrices

Definition

If $A = (a)$ is a 1×1-matrix, we let $\det(A) = a$. If $A = (A_{ij})$ is a $n \times n$-matrix with $n \geq 2$, then we define

$$\det(A) = \sum_{j=1}^{n} (-1)^{1+j} A_{1j} \det(\widetilde{A}_{1j}). \tag{‡}$$

Remark

The notation $|A|$ is sometimes used in place of $\det(A)$. Historically, the quantity

$$(-1)^{i+j} \det(\widetilde{A}_{ij})$$

is called the cofactor of the $(i,j)^{\text{th}}$-entry of A. Therefore the formula in (‡) is called the cofactor expansion of A along the first row.

Example

Example

$\det \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = A_{11} \det((A_{22})) - A_{12} \det((A_{21})) =$
$A_{11}A_{22} - A_{12}A_{21}$ which is the formula we gave for 2×2-matrices earlier.

Example

Let $A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & 1 \end{pmatrix}$. Then

$$\det(A) = 1 \cdot \det \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} - 2 \det \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} + 1 \cdot \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}$$

$$= -2 - 2(-1) + 3 = 3.$$

Lots of Effort

Remark

Of course, 4×4-example would require computing four 3×3-matrices each one of which would require three 2×2-matrices each requiring two multiplications. More generally, computing the determinant of a 25×25-matrix using the definition requires on the order of $25! = 15,511,210,043,330,985,984,000,000$ operations. At one trillion operations a second, it would take almost 500,000 years to compute. We may need a better way.

Zeros Are Our Friends: Part I

Lemma

Let I_n be the $n \times n$-identity matrix. Then $\det(I_n) = 1$.

Proof.

We have $\det(I_n) = \sum_{j=1}^{n} (-1)^{1+j} (I_n)_{1j} \det(\widetilde{I}_{n1j}) = \det(I_{n-1})$. Since $\det(I_2) = 1$, we can use induction. □

Multi-Linear

Let $r_1, \dots, r_{k-1}, r_{k+1}, \dots, r_n$ be $n-1$ vectors in $\mathbf{F}^n$. Then for each $u \in \mathbf{F}^n$ we can form a $n \times n$ matrix

$$A(u) = \begin{pmatrix} r_1 \\ \vdots \\ r_{k-1} \\ u \\ r_{k+1} \\ \vdots \\ r_n \end{pmatrix}.$$

Then we get a function $T_k : \mathbf{F}^n \to \mathbf{F}$ defined by $T_k(u) = \det(A(u))$.

Theorem

The function T_k defined above is a linear map from $\mathbf{F}^n$ to $\mathbf{F}$. That is, $\det : M_{n\times n}(\mathbf{F}) \to \mathbf{F}$ is a linear function of each of its rows when the other rows are held fixed.

Proof

Proof.

The result is immediate if $n = 1$. We will proceed by induction and assume the result for $(n-1) \times (n-1)$-matrices for $n \geq 2$ and consider a $n \times n$-matrix A with rows $r_1, \ldots, r_n$. Assume that $r_k = u + kv$ for $u, v \in \mathbf{F}^n$ and $k \in \mathbf{F}$. We need to prove that

$$\det(A) = \det(B) + k \det(C)$$

where B is obtained from A by replacing r_k by u and C is obtained from A by replacing r_k with v. If $k = 1$, the result follows easily from the definition. So we can assume $k > 1$. □

Proof

Proof Continued.

Note that for $1 \leq j \leq n$. the rows of $\widetilde{A}_{1j}$, $\widetilde{B}_{1j}$, and $\widetilde{C}_{1j}$ are equal except for the $(k-1)^{\text{st}}$-row. The $(k-1)^{\text{st}}$-row of $\widetilde{A}_{1j}$ is

$$(u_1 + kv_1, \dots, u_{j-1} + kv_{j-1}, u_{j+1} + kv_{j+1}, \dots, u_n + kv_{n+1})$$

which is the $(k-1)^{\text{st}}$-row of $\widetilde{B}_{1j}$ plus k times the $(k-1)^{\text{st}}$-row of $\widetilde{C}_{1j}$. Thus by the inductive hypothesis

$$\det(\widetilde{A}_{1j}) = \det(\widetilde{B}_{1j}) + k\det(\widetilde{C}_{1j}).$$

Proof

Proof Continued.

Since $A_{1j} = B_{1j} = C_{1j}$, we have

$$\begin{aligned}
\det(A) &= \sum_{j=1}^{n} (-1)^{1+j} A_{1j} \det(\widetilde{A}_{1j}) \\
&= \sum_{j=1}^{n} (-1)^{1+j} A_{1j} \cdot \left[\det(\widetilde{B}_{1j}) + k \det(\widetilde{C}_{1j})\right] \\
&= \sum_{j=1}^{n} (-1)^{1+j} B_{1j} \det(\widetilde{B}_{1j}) + k \sum_{j=1}^{n} (-1)^{1+j} C_{1j} \det(\widetilde{C}_{1j}) \\
&= \det(B) + k \det(C)
\end{aligned}$$

as required. This completes the proof. □

Corollary

Suppose that A is a $n \times n$-matrix with a row of all zeros. Then $\det(A) = 0$*.*

Proof.

Suppose that the k^{th}-row of A is zero. Let $A(u)$ be the $n \times n$-matrix with the same rows as A except with the k^{th}-row replaced by u. Then we just proved that $T(u) = \det(A(u))$ is linear. Then $0 = T(0) = \det(A(0)) = \det(A)$. □

Break Time

Time for a break and questions.

Our First Major Result on Determinants

Theorem

If A is a $n \times n$-matrix, then we can compute $\det(A)$ *be expanding along any row; that is, for all $1 \le i \le n$,*

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} A_{ij} \det(\widetilde{A}_{ij}) \tag{$\dagger$}$$

Remark

Since $c_{ij} = (-1)^{i+j} \det(\widetilde{A}_{ij})$ is the cofactor of the $(i,j)^{\mathrm{th}}$-entry of A, $(\dagger)$ is called the cofactor expansion along the i^{th}-row of A.

First an Example

Example

Let $A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & 1 \end{pmatrix}$. Then expanding along the second row

$$\begin{aligned} \det(A) &= -1 \cdot \det \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} - 1 \det \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} - 1 \cdot \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \\ &= -1(1) - 1(-1) - (-3) = 3. \end{aligned}$$

Remark (The Checker Board)

When computing cofactor expansions of determinants, it is all about the sign of the cofactor! I use this model:

$$\begin{pmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Crucial Special Case

For the proof, we need to first establish the following special case. Unfortunately, the proof is quite technical.

Lemma (Technical Lemma)

Suppose that $B \in M_{n\times n}(\mathbf{F})$ *with* $n \geq 2$. *Suppose that the* i^{th}*-row of* B *is the standard basis vector* e_k *with* $1 \leq k \leq n$. *Then*

$$\det(B) = (-1)^{i+k} \det(\widetilde{B}_{ik}).$$

Proof.

You can verify the result yourself if $n = 2$. So we proceed by induction and assume that result is true for $(n-1)\times(n-1)$-matrices with $n \geq 3$. Now assume that B is a $n \times n$-matrix whose i^{th}-row is e_k. If $i = 1$, then the result follows immediately from the definition. So we assume $1 < i \leq n$.

Proof

Proof Continued.

For each $j \neq k$ $(1 \leq j \leq n)$, let C_{ij} be the $(n-2) \times (n-2)$-matrix obtained from B by deleting rows 1 and i as well as columns j and k. Therefore we can compute $\det(\widetilde{B}_{ik})$ using the definition as follows:

$$\begin{aligned}
\det(\widetilde{B}_{ik}) &= \sum_{j=1}^{n-1} (-1)^{1+j} (\widetilde{B}_{ik})_{1j} \det(\widetilde{(\widetilde{B}_{ik})_{1j}}) \\
&= \sum_{j=1}^{k-1} (-1)^{1+j} B_{1j} \det(C_{ij}) + \sum_{j=k}^{n-1} (-1)^{1+j} B_{1,j+1} \det(C_{i,j+1}) \\
&= \sum_{j<k} (-1)^{1+j} B_{1j} \det(C_{ij}) + \sum_{j>k} (-1)^{1+(j-1)} B_{1j} \det(C_{ij}).
\end{aligned}$$

Return

Proof

Proof.

Now we turn to $\det(\widetilde{B}_{1j})$ which is a $(n-1)\times(n-1)$-matrix to which that induction hypothesis applies. Notice that the $(i-1)^{\text{th}}$-row of $\widetilde{B}_{1j}$ is

$$\begin{cases} e_{k-1} & \text{if } j < k, \\ 0 & \text{if } j - k, \text{ and} \\ e_k & \text{if } j > k. \end{cases}$$

Using the induction hypothesis and the corollary,

$$\det(\widetilde{B}_{1j}) = \begin{cases} (-1)^{(i-1)+(k-1)} \det(C_{ij}) & \text{of } j < k, \\ 0 & \text{if } j = k, \text{ and} \\ (-1)^{(i-1)+k} \det(C_{ij}) & \text{if } j > k. \end{cases}$$

Proof

Proof Continued.

Therefore

$$\begin{aligned}
\det(B) &= \sum_{j=1}^{n} (-1)^{1+j} B_{1j} \det(\widetilde{B}_{1j}) \\
&= \sum_{j<k} (-1)^{1+j} B_{1j} \det(\widetilde{B}_{1j}) + \sum_{j>k}^{n} (-1)^{1+j} B_{1j} \det(\widetilde{B}_{1j}) \\
&= \sum_{j<k} (-1)^{1+j} B_{1j} \Big[(-1)^{(i-1)+(k-1)} \det(C_{ij})\Big] \\
&\qquad + \sum_{j>k} (-1)^{1+j} B_{1j} \Big[(-1)^{(i-1)+k} \det(C_{ij})\Big] \\
&= (-1)^{i+k} \Bigg[\sum_{j<k} (-1)^{1+j} B_{1j} \det(C_{ij}) \\
&\qquad\qquad + \sum_{j>k} (-1)^{1+(j-1)} B_{1j} \det(C_{ij})\Bigg] \\
&= (-1)^{i+k} \det(\widetilde{B}_{ik})
\end{aligned}$$

using our previous formula for $\det(\widetilde{B}_{ik})$. This is what we wanted to show. □

The Theorem

Theorem

If A is a $n \times n$-matrix, then we can compute $\det(A)$ be expanding along any row; that is, for all $1 \leq i \leq n$,

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} A_{ij} \det(\widetilde{A}_{ij})$$

Proof.

If $i = 1$, this is just the definition. Otherwise, write the i^{th}-row as $\sum_{j=1}^{n} A_{ij} e_j$. Let B_j be the matrix obtained from A by replacing its i^{th}-row with e_j. Then since det is a linear function of the rows, we can use our lemma to compute that

$$\det(A) = \sum_{j=1}^{n} A_{ij} \det(B_j) = \sum_{j=1}^{n} A_{ij} (-1)^{i+j} \det(\widetilde{(B_j)}_{ij})$$

$$= \sum_{j=1}^{n} (-1)^{i+j} A_{ij} \det(\widetilde{A}_{ij})$$

□

Enough

1. That is enough for today.