Math 24: Winter 2021
Lecture 17

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Let's Get Started

1. We should be recording.
2. Remember, it is more comfortable for me if you turn on your video so that I feel like I am talking to real people.
3. Our Midterm will be available Thursday after office hours and must be turned in by Saturday, February 20th, at 10pm. No exceptions. It will cover through today's lecture and hence all of §4.3 in the text.
4. This time you will have 3.5 hours with 30 minutes uploading time. You should plan ahead and block out a time to take the exam now.
5. But first, are there any questions from last time?

The Determinant for $n \times n$-Matrices

Definition

If $A = (a)$ is a 1×1-matrix, we let $\det(A) = a$. If $A = (A_{ij})$ is a $n \times n$-matrix with $n \geq 2$, then we define

$$\det(A) = \sum_{j=1}^{n} (-1)^{1+j} A_{1j} \det(\widetilde{A}_{1j}).$$

Theorem

The function $\det : M_{n \times n}(\mathbf{F}) \to \mathbf{F}$ *is a linear function of each of its rows when the other rows are held fixed.*

Theorem

If A is a $n \times n$-matrix, then we can compute $\det(A)$ *be expanding along any row; that is, for all* $1 \leq i \leq n$,

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} A_{ij} \det(\widetilde{A}_{ij})$$

Properties of the Determinant

Proposition

If $A \in M_{n\times n}(\mathbf{F})$ *has two identical rows, then* $\det(A) = 0$*.*

Proof.

This is easy to check if $n = 2$. We proceed by induction and assume the result for $(n-1)\times(n-1)$-matrices with $n \geq 3$. Suppose A is a $n \times n$-matrix whose r^{th} and s^{th} rows are identical. Since $n \geq 3$, we can pick a row, say the i^{th}-row, with i not equal to r or s. Then by our last theorem,

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} A_{ij} \det(\widetilde{A}_{ij}).$$

But now each $\widetilde{A}_{ij}$ has two identical rows. Then by our induction hypothesis, $\det(\widetilde{A}_{ij}) = 0$ for all j and $\det(A) = 0$. This completes the proof. □

Determinants and Elementary Row Operations

Proposition (Type 1 row operations)

Suppose $A \in M_{n\times n}(\mathbf{F})$ and that B is obtained from A by interchanging two rows. Then $\det(B) = -\det(A)$.

Proof.

Let $r_1, \dots, r_n$ be the rows of A and that B is obtained from A by interchanging rows i and k with $i < k$.

Thus $A = \begin{pmatrix} \vdots \\ r_i \\ \vdots \\ r_k \\ \vdots \end{pmatrix}$ and $B = \begin{pmatrix} \vdots \\ r_k \\ \vdots \\ r_i \\ \vdots \end{pmatrix}$.

Proof

Proof Continued.

But since the determinant is linear in each row, we can use the previous proposition to conclude that

$$0 = \det \begin{pmatrix} \vdots \\ r_i + r_k \\ \vdots \\ r_i + r_k \\ \vdots \end{pmatrix} = \det \begin{pmatrix} \vdots \\ r_i \\ \vdots \\ r_i + r_k \\ \vdots \end{pmatrix} + \det \begin{pmatrix} \vdots \\ r_k \\ \vdots \\ r_i + r_k \\ \vdots \end{pmatrix}$$

$$= \det \begin{pmatrix} \vdots \\ r_i \\ \vdots \\ r_i \\ \vdots \end{pmatrix} + \det \begin{pmatrix} \vdots \\ r_i \\ \vdots \\ r_k \\ \vdots \end{pmatrix} + \det \begin{pmatrix} \vdots \\ r_k \\ \vdots \\ r_i \\ \vdots \end{pmatrix} + \det \begin{pmatrix} \vdots \\ r_k \\ \vdots \\ r_k \\ \vdots \end{pmatrix}$$

$$= 0 + \det(A) + \det(B) + 0.$$

Hence $\det(B) = -\det(A)$ as claimed. □

Type 3 Elemenary Row Operations

Proposition (Type 3 Row Operations)

Suppose $A \in M_{n\times n}(\mathbf{F})$ and B is obtained from A by adding a multiple of one row to another row. Then $\det(B) = \det(A)$.

Proof.

Let $r_1, \dots, r_n$ be the rows of A and let $A(u)$ be the matrix with rows $r_1, \dots, r_{k-1}, u, r_{k+1}, \dots, r_n$. Then $u \mapsto \det(A(u))$ is linear from $\mathbf{F}^n$ to $\mathbf{F}$. Suppose $B = A(r_k + cr_i)$ with $i \neq k$. Then

$$\det(B) = \det(A(r_k + cr_i)) = \det(A(r_k)) + c\det(A(r_i)). \quad (\ddagger)$$

But $A(r_k) = A$ and $A(r_i)$ has two identical rows. Hence $(\ddagger)$ implies $\det(B) = \det(A)$ as claimed. □

Example

Example

Show that $\det(C) = 10$ where $C = \begin{pmatrix} 2 & 0 & 4 & 2 \\ 1 & 1 & 0 & 1 \\ 0 & 2 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix}$.

Solution

Well, there is no good row to expand along that doesn't involve at least three 3×3*-determinants. But it is not so hard to use just type 3 elementary row operations to transform* C *into* $B = \begin{pmatrix} 2 & 0 & 4 & 2 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 6 & 1 \\ 0 & 0 & 0 & \frac{5}{6} \end{pmatrix}$. *Since* $\det(C) = \det(B)$, *we just have to compute* $\det(B)$. *But this is easy! Expand along the bottom rows to get*

$$\det(B) = \frac{5}{6} \det \begin{pmatrix} 2 & 0 & 4 \\ 0 & 1 & -2 \\ 0 & 0 & 6 \end{pmatrix} = \left(\frac{5}{6}\right) \cdot 6 \cdot \det \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = 10.$$

General Procedure

Remark

In general, we can transform any $n \times n$-matrix A into an upper-triangular matrix B using only type 1 and type 3 elementary row operations. Then $\det(A) = (-1)^r \det(B)$ where r is the number of type 1 operations used. It is a homework exercise to realize that $\det(B)$ is just the product of its diagonal entries.

Break Time

Let's take a break and see if there are any questions.

Theorem

Suppose that $A \in M_{n\times n}(\mathbf{F})$.

1. *If* B *is obtained from* A *by a type 1 elementary row operation (interchanging two rows), then* $\det(B) = -\det(A)$.
2. *If* B *is obtained from* A *by a type 2 elementary row operation, say multiplying a row by the scalar* c*, then* $\det(B) = c\det(A)$.
3. *If* B *is obtained from* A *by a type 3 elementary row operation (adding a multiple of one row to another row), then* $\det(B) = \det(A)$.

Proof.

We proved item (1) and (3) prior to the break. Item (2) is immediate as the determinate is a linear function of each row. □

Elementary Matrices

Corollary

Suppose that E is an elementary $n \times n$-matrix.

1. *If E is type 1, then $\det(E) = -1$.*
2. *If E is type 2, then $\det(E) = c$ where c is the nonzero scalar used to create E.*
3. *If E is type 3, then $\det(E) = 1$.*

In all cases, $\det(E) \neq 0$.

Proof.

The matrix E is obtained from I_n by performing the corresponding elementary row operation and $\det(I_n) = 1$. □

Invertible Matrices

Corollary

If $A \in M_{n\times n}(\mathbf{F})$ is not invertible, then $\det(A) = 0$.

Proof.

Note that if B is obtained from A by an elementary row operation and if $\det(A) = 0$, then $\det(B) = 0$. Since elementary row operations are reversible, we have $\det(A) = 0$ if and only if $\det(B) = 0$.

If A is not invertible, then $\operatorname{rank}(A) < n$. Thus if B is the reduced row echelon form of A, then $\operatorname{rank}(B) = \operatorname{rank}(A) < n$ and B has at least one row of all zeros. Thus $\det(B) = 0$. But since B is obtained from A by a sequence of elementary row operations, this implies $\det(A) = 0$. $\square$

An Amazing Result

Theorem

Suppose that A and B are $n \times n$-matrices. Then

$$\det(AB) = \det(A)\det(B).$$

Proof.

We start by assuming that A is an elementary matrix. Suppose that A is type 1. Then $C = AB$ is the matrix obtained from B by interchanging two rows. Hence

$$\det(AB) = -\det(B) = (-1)\det(B) = \det(A)\det(B).$$

A similar argument applies with A is type 2 or type 3.

If A is not invertible, then $\det(A) = 0$ and $\operatorname{rank}(A) < n$. But then $\operatorname{rank}(AB) \leq \operatorname{rank}(A) < n$ and $\det(AB) = 0$. Again,

$$\det(AB) = 0 = 0 \cdot \det(B) = \det(A)\det(B).$$

Proof

Proof Continued.

If A is invertible, then we know that $A = E_m \cdots E_1$ is a product of elementary matrices. Then, using the first part of the proof,

$$\begin{aligned}
\det(AB) &= \det(E_m \cdots E_1 B) \\
&= \det(E_m)\det(E_{m-1} \cdots E_1 B) \\
&\vdots \\
&= \det(E_m) \cdots \det(E_1)\det(B) \\
&= \det(E_m \cdots E_1)\det(B) \\
&= \det(A)\det(B).
\end{aligned}$$

So $\det(AB) = \det(A)\det(B)$ in all cases. □

Invertible Matrices

Corollary

A $n \times n$*-matrix* A *is invertible if and only if* $\det(A) \neq 0$*. In that case,* $\det(A^{-1}) = \frac{1}{\det(A)} = \det(A)^{-1}$.

Proof.

We already saw that if A is not invertible, then $\det(A) = 0$. But if A is invertible, then

$$1 = \det(I_n) = \det(AA^{-1}) = \det(A)\det(A^{-1}).$$

Hence $\det(A) \neq 0$ and $\det(A^{-1}) = \det(A)^{-1}$. □

Break Time

Time for a break and some questions.

The Transpose of an Elementary Matrix

Lemma

If E is a $n \times n$-elementary matrix, then E^t is an elementary matrix of the same type. In particular, $\det(E) = \det(E^t)$.

Sketch of the Proof.

If E is type 1 or type 2, then it is easy to check that E is symmetric; that is, $E^t = E$. If E is type three, say E adds c times row i to row j, then you can check that E^t adds c time row j to row i. In particular, E^t is type three and $\det(E^t) = 1 = \det(E)$. □

The Determinant of the Transpose

Theorem

If $A \in M_{n\times n}(\mathbf{F})$, then $\det(A^t) = \det(A)$.

Proof.

If A is invertible, then $A^{-1}A = I_n$. But then $A^t(A^{-1})^t = I_n^t = I_n$. Hence A^t is invertible and $(A^t)^{-1} = (A^{-1})^t$. Since $(A^t)^t = A$, we see that A is invertible if and only if A^t is. Thus if A is not invertible then $\det(A) = 0 = \det(A^t)$.

But if A is invertible, then $A = E_m \cdots E_1$ is a product of elementary matrices. Then $A^t = E_1^t \cdots E_m^t$. Thus

$$\begin{aligned}\det(A) &= \det(E_m)\cdots\det(E_1) = \det(E_m^t)\cdots\det(E_1^t) \\ &= \det(E_1^t)\cdots\det(E_m^t) = \det(E_1^t\cdots E_m^t) \\ &= \det(A^t).\end{aligned}$$

□

Columns Become Rows

Theorem

Suppose that $A = (A_{ij}) \in M_{n\times n}(\mathbf{F})$*. Then for all* $1 \le i \le n$ *and* $1 \le j \le n$*, we have*

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} A_{ij} \det(\widetilde{A}_{ij}) \quad \textit{and}$$

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} A_{ij} \det(\widetilde{A}_{ij}).$$

Proof.

The first equation if just our bit theorem from last time that we can expand along any row. The second equation is just applying the first to $\det(A^t)$. □

Column Operations

Theorem

Let $A \in M_{n\times n}(\mathbf{F})$.

1. *If B is obtained from A by interchanging two columns of A, then $\det(B) = -\det(A)$.*
2. *If B is obtained from A by multiplying one column by the scalar c, then $\det(B) = c\det(A)$.*
3. *If B is obtained from A by adding a multiple of one column to another column, then $\det(B) = \det(A)$.*

Proof.

Apply row operations to A^t. For example, if B is obtained from A by interchanging two columns, then B^t is obtained from A^t by interchanging two rows. Thus
$\det(B) = \det(B^t) = -\det(A^t) = -\det(A)$. □

Linear in Each Column

Corollary

The function det : $M_{n \times n}(\mathbf{F}) \to \mathbf{F}$ *is linear in each column if the other columns are held fixed.*

An Example

Example

Recall that $M \in M_{n\times n}(\mathbf{F})$ is called skew-symmetric if $M^t = -M$. For example, $M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Note that $\det(M) = \det(M^t) = \det(-M) = (-1)^n \det(M)$. Thus if n is odd, and $\mathbf{F}$ does not have characteristic 2, then $\det(M) = 0$ and M is not invertible.

Enough

1. That is enough for today.