Math 24: Winter 2021
Lecture 18

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Wednesday, February 17, 2021

Let's Get Started

1. We should be recording.
2. Remember, it is more comfortable for me if you turn on your video so that I feel like I am talking to real people.
3. Our Midterm will be available Thursday after office hours and must be turned in by Saturday, February 20th, at 10pm. No exceptions. It will cover through §4.3 in the text.
4. This time you will have 3.5 hours with 30 minutes uploading time. You should plan ahead and block out a time to take the exam now.
5. But first, are there any questions from last time?

Diagonalizable Transformations

Definition

A linear operator T on a finite-dimensional vector space V is said to be diagonalizable if there is an ordered basis β for V such that $[T]_\beta$ is a diagonal matrix.

Remark

Suppose that $\beta = \{ v_1, \dots, v_n \}$ is an ordered basis such that $[T]_\beta = \big[[T(v_1)]_\beta \ \cdots \ [T(v_n)]_\beta \big]$ is diagonal. That is,

$$\big[[T(v_1)]_\beta \ \cdots \ [T(v_n)]_\beta \big] = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

Then for each $1 \le k \le n$, $[T(v_k)]_\beta = \lambda_k e_k$ where $\{ e_1, \dots, e_n \}$ is the standard ordered basis for $\mathbf{F}^n$. That means $T(v_k) = \lambda_k v_k$ for all k.

Conversely, if $\beta = \{ v_1, \dots, v_n \}$ is an ordered basis for V such that $T(v_k) = \lambda_k v_k$ for all k, then $[T]_\beta$ is diagonal just as above.

25 Dollar Words

Definition

Suppose that T is a linear operator on a vector space V. Then a nonzero vector $v \in V$ is called an eigenvector for T if there is a scalar $\lambda \in \mathbf{F}$ such that $T(v) = \lambda v$. We call λ the eigenvalue corresponding the the eigenvector v.

If $A \in M_{n\times n}(\mathbf{F})$, then we call a nonzero $v \in \mathbf{F}^n$ an eigenvector for A if it is an eigenvector for L_A so that $Av = \lambda v$ for some scalar λ. Again, we call λ the eigenvalue corresponding to the eigenvector v.

Our First Eigen Theorem

We can summarize the previous discussion in a useful theorem.

Theorem

A linear operator T on a finite-dimensional vector space V is diagonalizable if and only if there is an ordered basis for V consisting of eigenvectors for T. If $\beta = \{ v_1, \dots, v_n \}$ is an ordered basis of eigenvectors for T, then $[T]_\beta$ is the diagonal matrix $D = (D_{ij})$ where

$$D_{ij} = \begin{cases} \lambda_i & \text{if } i = j, \text{ and} \\ 0 & \text{if } i \neq j \end{cases}$$

where λ_i is the eigenvalue corresponding to the eigenvector v_i.

Matrices

Corollary

A matrix $A \in M_{n\times n}(\mathbf{F})$ is diagonalizable if and only if there is an ordered basis of $\mathbf{F}^n$ consisting of eigenvectors for A. If $\beta = \{ v_1, \dots, v_n \}$ is an ordered basis of eigenvectors for A and if $Q = [v_1 \ \cdots \ v_n]$ is the $n \times n$-matrix whose j^{th}-column is v_j, then $D = Q^{-1}AQ$ is a diagonal matrix such that D_{ii} is the eigenvalue corresponding to v_i. In particular, A is diagonalizable if and only if it is similar to a diagonal matrix.

Proof.

The first statement follows from the previous theorem applied to $T = L_A$. If $\beta = \{ v_1, \dots, v_n \}$ is a basis of eigenvectors such that $Av_i = D_{ii}v_i$, then $[L_A]_\beta = D$ as in the statement of the corollary. But $Q = [I]_\beta^\sigma$ where σ is the standard ordered basis, and

$$D = [L_A]_\beta = [I]_\sigma^\beta [L_A]_\sigma^\sigma [I]_\beta^\sigma = Q^{-1}AQ.$$

Proof

Proof Continued.

Conversely, if Q is invertible and $D = Q^{-1}AQ$ is diagonal, then the columns of Q, say $\{ v_1, \dots, v_n \}$, are an ordered basis for $\mathbf{F}^n$ and $Q = [I]_\beta^\sigma$. Furthermore,

$$[L_A]_\beta^\beta = [I]_\sigma^\beta [L_A]_\sigma^\sigma [I]_\beta^\sigma = Q^{-1}AQ = D.$$

Thus $L_A(v_i) = Av_i = D_{ii}v_i$ and β is a basis of eigenvectors for A. □

Examples of Eigenvectors

Example

Let $A = \begin{pmatrix} 7 & -10 \\ 5 & -8 \end{pmatrix}$, $v_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, and $v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Then

$$L_A(v_1) = Av_1 = \begin{pmatrix} 7 & -10 \\ 5 & -8 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} = 2v_1$$

and

$$L_A(v_2) = Av_2 = \begin{pmatrix} 7 & -10 \\ 5 & -8 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ -3 \end{pmatrix} = -3v_2.$$

Since $\beta = \{ v_1, v_2 \}$ is an (ordered) basis, both L_A and A are diagonalizable. Furthermore, if $Q = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = [I]_\beta^\sigma$, then

$$Q^{-1}AQ = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix} = [L_A]_\beta.$$

Diagonalizability is Special

Example

Let $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Note that $A^2 = O$. If A—or equivalently L_A—were diagonalizable, then there would be a diagonal matrix D such that $D = Q^{-1}AQ$. But then
$D^2 = (Q^{-1}AQ)(Q^{-1}AQ) = Q^{-1}A^2Q = O$. But if $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, then $D^2 = \begin{pmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{pmatrix}$. So $D^2 = O$ implies $D = O$. But then $A = QDQ^{-1}$ would be the zero matrix! So A is not diagonalizable.

Rotations

Example

Let $T_\theta : \mathbf{R}^2 \to \mathbf{R}^2$ be rotation by θ radians about the origin. Then as in Example 2 in §2.1, $T_\theta = L_{A_\theta}$ where $A_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$. Now if $0 < \theta < \pi$, then it is clear from geometric considerations that T_θ has no eigenvectors! (Pictures are key here.)

Example

The previous example shows that $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ has no eigenvectors over $\mathbf{F} = \mathbf{R}$. (Here $\theta = \frac{\pi}{2}$ and remember that $T_{\frac{\pi}{2}}$ is a linear operator on the real-vector space $\mathbf{R}^2$.) But we could consider $A \in M_{2\times 2}(\mathbf{C})$. Then $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} i \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ i \end{pmatrix} = i\begin{pmatrix} i \\ 1 \end{pmatrix}$ and $v = \begin{pmatrix} i \\ 1 \end{pmatrix}$ is an eigenvector with eigenvalue i. Similarly, $\begin{pmatrix} -i \\ 1 \end{pmatrix}$ is an eigenvector with eigenvalue $-i$ and over the complex numbers A is diagonalizable with $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = Q^{-1}AQ$ with $Q = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}$. Fortunately, we are going to be working mostly over the reals. But we will need to think about the complex case from time to time.

Infinite Dimensions

Example

Now let $V = C^\infty(\mathbf{R})$ the subspace of the real-vector space $\mathscr{F}(\mathbf{R},\mathbf{R})$ of functions from $\mathbf{R}$ to $\mathbf{R}$ which have derivatives of all orders at every point. Define a linear operator T on $C^\infty(\mathbf{R})$ by $T(f) = f'$. What are the eigenvectors and eigenvalues of T?

Solution

If f is an eigenvector, then $f' = \lambda f$ for some $\lambda \in \mathbf{R}$. But then we are supposed to know from back in the day that this happens if and only if $f(x) = Ce^{\lambda x}$ for some constant $C \in \mathbf{R}$ such that $C \neq 0$. (Remember that eigenvector cannot be equal to 0_V!) Therefore every $\lambda \in \mathbf{R}$ is an eigenvalue for T and the eigenvectors with eigenvalue λ, or eigenfunctions if you like, are multiples of $f(x) = e^{\lambda x}$. (Thus the nonzero constant functions are the eigenfunctions for $\lambda = 0$.)

Break Time

Time for a break and some questions.

Eigenvalues are Rare

Theorem

Suppose that $A \in M_{n\times n}(\mathbf{F})$. Then $\lambda \in \mathbf{F}$ is an eigenvalue for A if and only if $\det(A - \lambda I_n) = 0$.

Proof.

Suppose that λ is an eigenvalue for A. Then there is a nonzero vector $v \in \mathbf{F}^n$ such that $Av = \lambda v$, or equivalently, $(A - \lambda I_n)v = 0$. Since $v \neq 0_V$, this implies that $A - \lambda I_n$ is not invertible and hence $\det(A - \lambda I_n) = 0$. Conversely if $\det(A - \lambda I_n) = 0$, then $A - \lambda I_n$ is not invertible and there is a nontrivial solution v to $(A - \lambda I_n)x = 0$ and then v is an eigenvector with eigenvalue λ. □

Example

Example

Let $A = \left(\begin{smallmatrix} 7 & -10 \\ 5 & -8 \end{smallmatrix}\right)$. Then $\det(A - \lambda I_2) = \det\left(\begin{smallmatrix} 7-\lambda & -10 \\ 5 & -8-\lambda \end{smallmatrix}\right) = (\lambda - 7)(\lambda + 8) + 50 = \lambda^2 + \lambda - 6 = (\lambda + 3)(\lambda - 2)$. It follows that the only eigenvalues of A are just $\lambda = -3$ and $\lambda = 2$. Of course, this was the example we looked at earlier.

An Observation

Proposition

Suppose $A \in M_{n\times n}(\mathbf{F})$. *Then* $p(\lambda) = \det(A - \lambda I_n)$ *is a polynomial in* $\mathrm{P}_n(\mathbf{F})$ *of the form*

$$p(\lambda) = (-1)^n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0.$$

In particular, p *has degree* n *and leading coefficient* $(-1)^n$.

For the proof, we need a little lemma.

Lemma

Let $A(\lambda)$ *be a* $n \times n$ *matrix with entries* $A_{ij}(\lambda) \in \mathrm{P}_1(\mathbf{F})$. *Then* $q(\lambda) = \det\big(A(\lambda)\big) \in \mathrm{P}_n(\mathbf{F})$.

Proof.

This is clear if $n = 1$, so let $n \geq 2$ and assume the result for $(n-1) \times (n-1)$-matrices. Then

$$\det(A(\lambda)) = \sum_{j=1}^{n} (-1)^{1+j} A_{1j}(\lambda) \det\big(\widetilde{(A(\lambda))}_{1j}\big).$$

The result follows since $\det\big(\widetilde{(A(\lambda))}_{1j}\big) \in \mathrm{P}_{n-1}(\mathbf{F})$ by assumption. □

Proof of the Proposition

The proposition is easy to check if $n \leq 2$. So assume $n \geq 3$ and that the result is true for $(n-1) \times (n-1)$-matrices. Then

$$\begin{aligned} p(\lambda) &= \det\big(A - \lambda I_n\big) \\ &= (A_{11} - \lambda)\det(\widetilde{A}_{11} - \lambda I_{n-1}) + \sum_{j=2}^{n} (-1)^{1+j} A_{1j} \det\big((\widetilde{A - \lambda I_n})_{1j}\big) \end{aligned}$$

which, by the induction hypotheses, is

$$= (A_{11} - \lambda)\big((-1)^{n-1}\lambda^{n-1} + \cdots + b_0\big) + \sum_{j=2}^{n} (-1)^{1+j} A_{1j} \det\big((\widetilde{A - \lambda I_n})_{1j}\big)$$

$$= (-1)^n \lambda^n + p_0(\lambda) + \sum_{j=2}^{n} p_j(\lambda)$$

where $p_0 \in \mathsf{P}_{n-1}(\mathbf{F})$. Since each $p_j \in \mathsf{P}_{n-1}(\mathbf{F})$ by the lemma. The result now follows easily.

Characteristic Polynomial

Definition

If $A \in M_{n\times n}(\mathbf{F})$, then $p(\lambda) = \det(A - \lambda I_n)$ is called the characteristic polynomial of A.

Proposition

If $A \in M_{n\times n}(\mathbf{F})$, then A has at most n eigenvalues.

Proposition

Our technical proposition implies that the characteristic polynomial $p(\lambda) = \det(A - \lambda I_n)$ is a polynomial of degree n. Hence p has at most n roots. (We call λ a root of p if and only if $p(\lambda) = 0$.) But our earlier theorem implies that λ is an eigenvalue if and only if λ is a root of the characteristic polynomial.

Finding Eigenvectors

Remark

Let $A \in M_{n\times n}(\mathbf{F})$. If λ is a root of the characteristic polynomial $p(\lambda) = \det(A - \lambda I_n)$, then λ is an eigenvalue. The eigenvectors with eigenvalue λ are exactly the nontrivial solutions to $(A - \lambda I_n)x = 0$.

Example

Try to diagonalize $A = \left(\begin{smallmatrix} 1 & 2 \\ 3 & 2 \end{smallmatrix}\right)$.

Solution

First compute the characteristic polynomial $p(\lambda) = \det(A - \lambda I_n) = \det\left(\begin{smallmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{smallmatrix}\right) = (\lambda - 1)(\lambda - 2) - 6 = \lambda^2 - 3\lambda - 4 = (\lambda + 1)(\lambda - 4)$.

Solution (Solution Continued)

Therefore the eigenvalues are just $\lambda = -1$ *and* $\lambda = 4$. *To find eigenvalues, we consider* $(A - \lambda I_2)x = 0$. *Since* $A + I_2 = \begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix}$ *is row equivalent to* $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, *the solutions to* $(A + I_2)x = 0$ *are the same as the solutions to* $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} x = 0$. *(Why!?! Consider the augmented matrices.) Thus all the eigenvectors are (nonzero) multiples of* $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$. *On the other hand* $A - 4I_2 = \begin{pmatrix} -3 & 2 \\ 3 & -2 \end{pmatrix}$ *is row equivalent to* $\begin{pmatrix} -3 & 2 \\ 0 & 0 \end{pmatrix}$, *so the solutions to* $(A - 4I_2)x = 0$ *are the same as for* $\begin{pmatrix} -3 & 2 \\ 0 & 0 \end{pmatrix} x = 0$. *Hence the eigenvectors with eigenvalue 4 are all (nonzero) multiples of* $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$. *Hence* $\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix} \}$ *is a basis of eigenvectors for* A *and if* $Q = \begin{pmatrix} -1 & 2 \\ 1 & 3 \end{pmatrix}$ *then*

$$Q^{-1}AQ = \begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix}.$$

Note that there are many possible choices for Q *here!*

A Bigger Example

Example

Now let $A = \begin{pmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{pmatrix}$. Then

$p(\lambda) = \det(A - \lambda I_3) = \det \begin{pmatrix} 1-\lambda & 3 & 3 \\ -3 & -5-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{pmatrix}$ is given by

$$p(\lambda) = (1-\lambda)\det\begin{pmatrix} -5-\lambda & -3 \\ 3 & 1-\lambda \end{pmatrix} - 3\det\begin{pmatrix} -3 & -3 \\ 3 & 1-\lambda \end{pmatrix} + 3\det\begin{pmatrix} -3 & -5-\lambda \\ 3 & 3 \end{pmatrix}$$

$$\begin{aligned} &= (1-\lambda)\big[(\lambda+5)(\lambda-1)+9\big] - 3(3(\lambda-1)+9) \\ &\qquad + 3(-9+3(\lambda+5)) \\ &= (1-\lambda)(\lambda^2+4\lambda+4) - 9(\lambda+2) + 9(\lambda+2) \\ &= -(\lambda-1)(\lambda+2)^2. \end{aligned}$$

Example (Continued)

Thus the eigenvalues are $\lambda = 1$ and $\lambda = -2$. To find the eigenvectors for $\lambda = -2$ we consider

$$A + 2I_3 = \begin{pmatrix} 3 & 3 & 3 \\ -3 & -3 & -3 \\ 3 & 3 & 3 \end{pmatrix} \underbrace{\sim}_{\text{row equivalent}} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence the eigenvectors for $\lambda = -2$ are spanned by $v_1 = (-1, 0, 1)$ and $v_2 = (1, -1, 0)$.

Similarly,

$$A - I_3 = \begin{pmatrix} 0 & 3 & 3 \\ -3 & -6 & -3 \\ 3 & 3 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus the $\lambda = 1$ eigenvectors are nonzero multiples of $v_3 = (1, -1, 1)$.

Example

Example (Continued)

Now we need to see that $\beta = \{ v_1, v_2, v_3 \}$ is a basis. But if Q is the matrix with columns v_1, v_2, and v_3, then

$$\det(Q) = \det \begin{pmatrix} -1 & -1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix} = \det \begin{pmatrix} -1 & -1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$
$$= \det\left(\begin{smallmatrix} -1 & -1 \\ 0 & 1 \end{smallmatrix}\right) = -1 \neq 0.$$

Hence β is a basis of eigenvectors and $Q^{-1}AQ = \left(\begin{smallmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$.

Enough

1. That is enough for today.