Math 24: Winter 2021
Lecture 19

Dana P. Williams

Dartmouth College

Friday, February 19, 2021

Let's Get Started

1. We should be recording.
2. Remember, it is more comfortable for me if you turn on your video so that I feel like I am talking to real people.
3. Our Midterm must be turned in by Saturday, February 20th, at 10pm. No exceptions. It covers through §4.3 in the text.
4. This time you have 3.5 hours with 30 minutes uploading time. You should plan ahead and block out a time to take the exam now.
5. But first, are there any questions from last time?

Review

Definition

A linear operator T on a finite-dimensional vector space V is said to be diagonalizable if there is an ordered basis β for V such that $[T]_\beta$ is a diagonal matrix.

Definition

Suppose that T is a linear operator on a vector space V. Then a nonzero vector $v \in V$ is called an eigenvector for T if there is a scalar $\lambda \in \mathbf{F}$ such that $T(v) = \lambda v$. We call λ the eigenvalue corresponding the the eigenvector v.

If $A \in M_{n\times n}(\mathbf{F})$, then we call a nonzero $v \in \mathbf{F}^n$ an eigenvector for A if it is an eigenvector for L_A so that $Av = \lambda v$ for some scalar λ. Again, we call λ the eigenvalue corresponding to the eigenvector v.

Review

Theorem

A linear operator T on a finite-dimensional vector space V is diagonalizable if and only if there is an ordered basis for V consisting of eigenvectors for T. If $\beta = \{ v_1, \dots, v_n \}$ is an ordered basis of eigenvectors for T, then $[T]_\beta$ is the diagonal matrix $D = (D_{ij})$ where

$$D_{ij} = \begin{cases} \lambda_i & \text{if } i = j, \text{ and} \\ 0 & \text{if } i \neq j \end{cases}$$

where λ_i is the eigenvalue corresponding to the eigenvector v_i.

Review

Corollary

A matrix $A \in M_{n\times n}(\mathbf{F})$ is diagonalizable if and only if there is an ordered basis of $\mathbf{F}^n$ consisting of eigenvectors for A. If $\beta = \{ v_1, \ldots, v_n \}$ is an ordered basis of eigenvectors for A and if $Q = [v_1 \ \cdots \ v_n]$ is the $n \times n$-matrix whose j^{th}-column is v_j, then $D = Q^{-1}AQ$ is a diagonal matrix such that D_{ii} is the eigenvalue corresponding to v_i. In particular, A is diagonalizable if and only if it is similar to a diagonal matrix.

Review

Theorem

Suppose that $A \in M_{n\times n}(\mathbf{F})$. Then $\lambda \in \mathbf{F}$ is an eigenvalue for A if and only if $\det(A - \lambda I_n) = 0$.

Definition

If $A \in M_{n\times n}(\mathbf{F})$, then $p(\lambda) = \det(A - \lambda I_n)$ is called the characteristic polynomial of A.

Remark

If $A \in M_{n\times n}(\mathbf{F})$, then its characteristic polynomial has the form $p(\lambda) = (-1)^n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$ with each $a_k \in \mathbf{F}$. In particular, $p(\lambda) \in \mathrm{P}_n(\mathbf{F})$ and has degree equal to n.

Proposition

If $A \in M_{n\times n}(\mathbf{F})$, then A has at most n eigenvalues.

Determinants of Linear Operators

Remark

As we saw Wednesday, the key to finding the eigenvalues of a matrix, and hence its eigenvectors, is its characteristic polynomial obtained via the determinant. Suppose now that T is a linear operator on a finite-dimensional vector space V with $\dim(V) = n$. If β is an ordered basis for V, then $[T]_\beta$ is a $n \times n$-matrix so we can compute its determinant $\det\bigl([T]_\beta\bigr)$. If γ is another ordered basis for V, then the Change of Basis Theorem implies that $[T]_\gamma = Q^{-1}[T]_\beta Q$ for the change of coordinate matrix $[Q] = [I_V]_\gamma^\beta$. In particular, $[T]_\beta$ and $[T]_\gamma$ are similar. Then as you proved on homework, $\det\bigl([T]_\gamma\bigr) = \det\bigl([T]_\beta\bigr)$. This allows us to make the following definition.

Definition

If V is a finite-dimensional vector space and $T \in \mathcal{L}(V)$, then the determinant of T, written $\det(T)$, is given by $\det\bigl([T]_\beta\bigr)$ where β is any ordered basis for V.

Invertibility

Proposition

Suppose that V is a finite-dimensional vector space and $T \in \mathcal{L}(V)$. Then T is invertible if and only if $\det(T) \neq 0$.

Proof.

If β is any ordered basis for V, then we proved that T is invertible if and only if $[T]_\beta$ is. But $[T]_\beta$ is invertible if and only if $\det\bigl([T]_\beta\bigr) \neq 0$ and $\det\bigl([T]_\beta\bigr) = \det(T)$ by definition. $\square$

Characteristic Polynomial

Definition

If T is a linear operator on a finite-dimensional vector space V, then the characteristic polynomial of T is $p(\lambda) = \det(T - \lambda I_V)$.

Remark

If β is an ordered basis for V and $\dim(V) = n$, then we can "transfer" the problem of finding eigenvalues and eigenvectors for $T \in \mathcal{L}(V)$ to the corresponding problem for the matrix $A = [T]_\beta$ via our standard picture:

$$\begin{array}{ccc} V & \xrightarrow{\quad T \quad} & V \\ {\scriptstyle \varphi_\beta}\downarrow & & \downarrow{\scriptstyle \varphi_\beta} \\ \mathbf{F}^n & \xrightarrow{\quad L_A \quad} & \mathbf{F}^n \end{array}$$

where φ_β is the standard representation of V given by $\varphi_\beta(v) = [v]_\beta$.

Say What?

Example

For example, suppose that x is an eigenvector for $A = [T]_\beta$ with eigenvalue λ. That is, $Ax = \lambda x$. Let $v = \varphi_\beta^{-1}(x)$. Then $[T(v)]_\beta = [T]_\beta [v]_\beta = Ax = \lambda x = \lambda [v]_\beta = [\lambda v]_\beta$. Since φ_β is an isomorphism, this means $T(v) = \lambda v$. I leave it to you to check that the argument is reversible; that is, if v is an eigenvector for T with eigenvalue λ, then $\varphi_\beta(v) = [v]_\beta$ is an eigenvector for $A = [T]_\beta$ with eigenvalue λ.

Since $p(\lambda) = \det(T - \lambda I_V) = \det\big([T - \lambda I_V]_\beta\big) = \det\big([T]_\beta - \lambda I_n\big)$, the characteristic polynomial of T and $A = [T]_\beta$ are the same. Hence the eigenvalues for T are exactly the roots of its characteristic polynomial.

Example

Define $T : P_2(\mathbf{R}) \to P_2(\mathbf{R})$ be given by $T(p(x)) = p(x) + (1+x)p'(x) + p''(x)$. You can verify that T is a linear operator. Let $\beta = \{1, x, x^2\}$ be the standard basis for $P_2(\mathbf{R})$. Then

$$[T]_\beta = \big[[T(1)]_\beta \; [T(x)]_\beta \; [T(x^2]_\beta\big].$$

Since $T(1) = 1$, $T(x) = 2x + 1$ and $T(x^2) = x^2 + (1+x)2x + 2 = 3x^2 + 2x + 2$, we have

$$[T]_\beta = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}.$$

Example Continued

Example (Continued)

Since the matrix is upper triangular, the characteristic polynomial is $p(\lambda) = -(\lambda - 1)(\lambda - 2)(\lambda - 3)$ and the eigenvalues are $\lambda = 1, 2, 3$. To find the eigenvectors, we consider $A = [T]_\beta$ which has the same eigenvalues! We start with $\lambda = 1$ and consider the homogeneous system $(A - I_3)x = 0$. But

$$A - I_3 = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix} \underbrace{\sim}_{\text{row equivalent}} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus $(A - I_3)x = 0$ is equivalent to the system $x_2 = 0$ and $x_3 = 0$. Therefore the eigenvectors with eigenvalue $\lambda = 1$ are all nonzero multiples of $u_1 = (1, 0, 0)$. Next we consider

$$A - 2I_3 = \begin{pmatrix} -1 & 1 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence the eigenvectors with eigenvalue $\lambda = 2$ are all nonzero multiples of $u_2 = (1, 1, 0)$.

Example Continued

Example (Continued)

I will leave it to you to check that the eigenvectors with eigenvalue $\lambda = 3$ are all multiples of $u_3 = (2,2,1)$ and that $\{ u_1, u_2, u_3 \}$ are a basis in $\mathbf{R}^3$ of eigenvectors for A. Since φ_β^{-1} is an isomorphism, $\gamma = \{ 1, 1+x, 2+2x+x^2 \}$ is a basis for $\mathrm{P}_2(\mathbf{R})$ of eigenvectors for T. Moreover,

$$[T]_\gamma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

Break Time

Time for a break and some questions.

Eigenspaces

Definition

Suppose that $T \in \mathcal{L}(V)$ is a linear operator on a vector space V. If λ is an eigenvalue for T, then $E_\lambda = \{ v \in V : T(v) = \lambda v \}$ is called that λ-eigenspace for T. If $A \in M_{n\times n}(\mathbf{F})$ and λ is an eigenvalue for A, then the λ-eigenspace for A is just the λ-eigenspace for L_A.

Remark

Since $E_\lambda = \mathrm{N}(T - \lambda I_V)$, the λ-eigenspace is a subspace of V. The nonzero elements of E_λ are exactly the eigenvectors of T with eigenvalue λ.

Linear Independence

Remark

When trying to find a basis of eigenvectors, we need to worry about whether the eigenvectors we find are linearly independent. Our next result will make that work a lot easier as it will allow us to work one eigenvalue at a time. To show off, we will temporarily dispense with our standard assumption of finite-dimension.

Theorem

Suppose that V is a vector space and $T \in \mathcal{L}(V)$. Suppose that $\lambda_1, \dots, \lambda_k$ are distinct eigenvalues for T. For each $1 \le i \le k$, let S_i be a finite linear independent in E_{λ_i}. Then $S = S_1 \cup \cdots \cup S_k$ is linear independent.

Return

A Lemma

Remark

To make the proof a bit less messy, we will prove a special case first—namely the case where each S_i consists of a single vector.

Lemma

Suppose that V *is a vector space and* $T \in \mathcal{L}(V)$*. Suppose that* $\lambda_1, \dots, \lambda_k$ *are distinct eigenvalues of* T *and that* $v_i \in E_{\lambda_i}$ *for* $1 \le i \le k$*. If*

$$v_1 + v_2 + \cdots + v_k = 0_V,$$

then $v_i = 0_V$ *for all* $1 \le i \le k$*.*

Proof

Proof of the Lemma.

The result is trivial if $k = 1$. So we suppose that the result holds for k vectors and consider $k + 1$ vectors such that

$$v_1 + \cdots + v_k + v_{k+1} = 0_V$$

with each $v_i \in E_{\lambda_i}$ for $1 \leq i \leq k + 1$. Then

$$v_{k+1} = -v_1 - \cdots - v_k \tag{$*$}$$

If we apply T to both sides then

$$\lambda_{k+1} v_{k+1} = -\lambda_1 v_1 - \cdots - \lambda_k v_k.$$

But using $(*)$, we also have

$$\lambda_{k+1} v_{k+1} = -\lambda_{k+1} v_1 - \cdots - \lambda_{k+1} v_k$$

Proof

Proof Continued.

Therefore subtracting the last two equations gives

$$0_V = v_1' + \cdots + v_k'$$

where $v_i' = (\lambda_{k+1} - \lambda_i)v_i \in E_{\lambda_i}$. Hence our induction hypothesis implies that for all i

$$v_i' = 0_V = (\lambda_{k+1} - \lambda_i)v_i.$$

Since $\lambda_{k+1} - \lambda_i \neq 0$, we have $v_i = 0_V$ for $1 \leq i \leq k$. But then we also have $v_{k+1} = 0_V$. This completes the proof. $\square$

Proof of the Theorem

Proof of the Theorem.

Now the proof of the theorem.

Suppose that $S_i = \{ v_{i1}, \dots, v_{in_i} \}$ is a linearly independent subset of E_{λ_i} for each i and that $S = \bigcup_{i=1}^k S_i$. Suppose that

$$\sum_{i=1}^{k} \sum_{j=1}^{n_i} a_{ij} v_{ij} = 0_V.$$

We need to see that each a_{ij} must be 0. Let

$$v_i = \sum_{j=1}^{n_i} a_{ij} v_{ij}. \tag{$\dagger$}$$

Proof

Proof Continued.

Since E_{λ_i} is a subspace, each $v_i \in E_{\lambda_i}$ and we have

$$v_1 + \cdots + v_k = 0_V.$$

The lemma implies that $v_i = 0_V$ for each i. Since S_i is linearly independent, (†) implies that each $a_{ij} = 0$ for all i. This completes the proof. □

Distinct Eigenvalues

Corollary

Suppose that V is n-dimensional and that $T \in \mathcal{L}(V)$ has n-distinct eigenvalues. Then T is diagonalizable.

Proof.

Suppose that $\lambda_1, \dots, \lambda_n$ are the distinct eigenvalues of T. Let v_i be an eigenvector with eigenvalue λ_i. Then the previous theorem implies that $\beta = \{ v_1, \dots, v_n \}$ is linearly independent. Since $\dim(V) = n$, β is a basis of eigenvectors. Hence T is diagonalizable. $\square$

Example

Example

Let $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$. Then the characteristic polynomial of A is

$$\det(A - \lambda I_3) = \det \begin{pmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{pmatrix} =$$

$$-(\lambda - 1)\det \begin{pmatrix} \lambda - 1 & 1 \\ 1 & \lambda - 1 \end{pmatrix} = -(\lambda-1)\big[(\lambda-1)^2 - 1\big] =$$

$-(\lambda - 1)(\lambda^2 - 2\lambda)$ and the eigenvalues are 0, 1 and 2. Hence A is diagonalizable.

Converse

Remark

The converse of the previous result is not true. For example, at the end of yesterday's lecture we showed that $A = \begin{pmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{pmatrix}$.

Then $p(\lambda) = \det(A - \lambda I_3) = \det \begin{pmatrix} 1-\lambda & 3 & 3 \\ -3 & -5-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{pmatrix}$ had only two eigenvalues, $\lambda = 1$ and $\lambda = -2$. Nevertheless, we found a basis of eigenvectors and A is diagonalizable.

Of course, trivially, any diagonal matrix is diagonalizable no matter what entries it has on the diagonal. For example $A = I_n$ is diagonalizable for all n and has only one eigenvalue. You also showed on homework that $\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ is not diagonalizable.

Next week, we are going to have to look more deeply at criteria for diagonalizability.

Enough

1. That is enough for today.