Math 24: Winter 2021
Lecture 23

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Let's Get Started

1. We should be recording.
2. It is more fun for me if you turn your video on.
3. The final will be administered in a manner similar to the prelim and midterm exams. My plan is to release it on the first day of the final period, Saturday, March 13, at 8am EST, and require it to be completed by Monday, March 15 at 10pm EST. There will be what I hope is a comfortable window to complete the exam.
4. The final will cover the entire course and I hope to get to parts of Section 6.6. We probably will not be covering all of sections 6.2 – 6.6. You will only be responsible for what we conver in lecture. More details later.
5. But first, are there any questions from last time?

Review

Definition

Suppose that V is a vector space over $\mathbf{F}$ where $\mathbf{F}$ is either $\mathbf{R}$ or $\mathbf{C}$. Then an inner product on V is a function $\langle \cdot \,, \cdot \rangle : V \times V \to \mathbf{F}$ such that for all $x, y, z \in V$ and all $c \in \mathbf{F}$ we have

1. $\langle cx + y \,, z \rangle = c\langle x \,, z \rangle + \langle y \,, z \rangle$,
2. $\langle y \,, x \rangle = \overline{\langle x \,, y \rangle}$ where the bar is complex conjugation, and
3. $\langle x \,, x \rangle > 0$ if $x \neq 0$.

Example (Standard Inner Product)

If $V = \mathbf{F}^n$, then the standard inner product on $\mathbf{F}^n$ is given by

$$\langle x \,, y \rangle = \sum_{k=1}^{n} x_k \overline{y_k}.$$

This is the standard dot product in $\mathbf{R}^2$ and $\mathbf{R}^3$.

Review

Definition

If $A \in M_{m\times n}(\mathbf{F})$, then the conjugate transpose of A is the matrix $A^* \in M_{n\times n}(\mathbf{F})$ where $(A^*)_{ij} = \overline{A_{ji}}$. Of course $A^* = A^t$ if $\mathbf{F} = \mathbf{R}$.

Theorem

Suppose that V is an inner product space. Then for all $x, y, z \in V$ and $c \in \mathbf{F}$ we have

1. $\langle x , cy + z\rangle = \overline{c}\langle x , y\rangle + \langle x , z\rangle$,
2. $\langle x , 0_V\rangle = 0 = \langle 0_V , x\rangle$,
3. $\langle x , x\rangle = 0$ *if and only if* $x = 0_V$, *and*
4. *if* $\langle x , y\rangle = \langle x , z\rangle$ *for all* $x \in V$, *then* $y = z$.

Review

Definition

If V is an inner product space and $x \in V$, then we call $\|x\| = \sqrt{\langle x\,,\,x\rangle}$ the norm (or length) of x.

Theorem

Let V be an inner product space. Then for all $x, y \in V$ and $c \in \mathbf{F}$, we have

1. $\|cx\| = |c|\|x\|$,
2. $\|x\| \geq 0$ *and* $\|x\| = 0$ *if and only if* $x = 0_V$,
3. *(Cauchy-Schwarz Inequality)* $|\langle x\,,\,y\rangle| \leq \|x\|\|y\|$, *and*
4. *(Triangle Inequality)* $\|x + y\| \leq \|x\| + \|y\|$.

Orthogonality

Definition

Let V be an inner product space. They $x, y \in V$ are orthogonal (or perpendicular) if $\langle x , y\rangle = 0$. A subset $S \subset V$ is called orthogonal if any two distinct vectors in S are orthogonal. A vector $x \in V$ is called unit vector if $\|x\| = 1$. A subset $S \subset V$ is called orthonormal if it is an orthogonal set consisting of unit vectors.

Remark

Let $S = \{ v_1, \dots, v_n \}$ be a finite subset of an inner product space V. Then S is an orthonormal set if and only if

$$\langle v_i , v_j\rangle = \begin{cases} 1 & \text{if } i = j \text{, and} \\ 0 & \text{if } i \neq j. \end{cases}$$

Only So Much Room

Proposition

Suppose that V is an inner product space and $S = \{ v_1, \dots, v_n \}$ is an orthogonal set of nonzero vectors. Then S is linearly independent.

Proof.

Suppose that $a_1 v_1 + \cdots + a_n v_n = 0_V$. We need to see that this forces $a_k = 0$ for all k.

But for all k,

$$\begin{aligned} 0 = \langle 0_V \, , \, v_k \rangle &= \langle a_1 v_1 + \cdots + a_n v_n \, , \, v_k \rangle \\ &= a_1 \langle v_1 \, , \, v_k \rangle + \cdots + a_n \langle v_n \, , \, v_k \rangle \\ &= a_k \langle v_k \, , \, v_k \rangle = a_k \| v_k \|^2 . \end{aligned}$$

Since $v_k \neq 0_V$, $\| v_k \| \neq 0$. Thus $a_k = 0$. □

Even Better

Proposition

Suppose that $S = \{ v_1, \dots, v_n \}$ *is an orthogonal set of nonzero vectors in an inner product space* V*. If* $v \in \mathrm{Span}(S)$*, then*

$$v = \sum_{k=1}^{n} \frac{\langle v \, , \, v_k \rangle}{\| v_k \|^2} \cdot v_k .$$

In particular, if S *is an orthonormal subset and* $v \in \mathrm{Span}(S)$*, then*

$$v = \sum_{k=1}^{n} \langle v \, , \, v_k \rangle \cdot v_k . \qquad (\ddagger)$$

▸ Return

Proof

Proof.

By the previous proposition, S is a basis for $\text{Span}(S)$. Thus there are unique scalars a_j such that $v = \sum_{j=1}^{n} a_j v_j$. Then just as on the previous slide,

$$\langle v \, , \, v_k \rangle = a_k \cdot \langle v_k \, , \, v_k \rangle,$$

and the first equation follows. If S is orthonormal, then $\|v_k\| = 1$ for all k. Hence the second assertion follows as well. □

Orthogonal and Orthonormal Bases

Definition

If V is a finite-dimensional inner product space, then we call an ordered basis β an orthogonal basis if β is also an orthogonal set. If β is also orthonormal, we call β and orthonormal basis.

Example (Only The Best)

The standard ordered basis for $\mathbf{F}^n$ is an orthonormal basis.

Example

Example

It is not hard to see that $\gamma = \{ (1,1,1), (1,-1,0), (1,1,-2) \}$ is an orthogonal subset of $\mathbf{R}^3$. But then γ must be linearly independent. Since γ has three elements, it must be a basis—and hence an orthogonal basis. To get an orthonormal basis we just normalize: $\beta = \{ u_1, u_2, u_3 \} = \{ \frac{1}{\sqrt{3}}(1,1,1), \frac{1}{\sqrt{2}}(1,-1,0), \frac{1}{\sqrt{6}}(1,1,-2) \}$.

Let $v = (1,2,3)$. Then $v = a_1 u_1 + a_2 u_2 + a_3 u_3$. Then we have

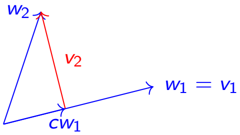
$$a_1 = \langle v \,,\, u_1 \rangle = \frac{6}{\sqrt{3}}, \quad a_2 = \langle v \,,\, u_2 \rangle = -\frac{1}{\sqrt{2}},$$
$$a_3 = \langle v \,,\, u_3 \rangle = -\frac{3}{\sqrt{6}}$$

Thus

$$(1,2,3) = 2(1,1,1) - \frac{1}{2}(1,-1,0) - \frac{1}{2}(1,1,-2).$$

Break Time

Time for a break and some questions.



Example

Suppose that V is an inner product space and that W is a subspace with a basis $\gamma = \{ w_1, w_2 \}$. We want an orthogonal basis $\{ v_1, v_2 \}$ for W.

Example (Continued)

The idea is to let $v_1 = w_1$ and then set $v_2 = w_2 - cw_1$ and choose c such that $\langle v_2 , v_1 \rangle = 0$. Then we want

$$0 = \langle v_2 , v_1 \rangle = \langle w_2 - cw_1 , w_1 \rangle = \langle w_2 , w_1 \rangle - c\langle w_1 , w_1 \rangle$$

or

$$c = \frac{\langle w_2 , w_1 \rangle}{\|w_1\|^2} \quad \text{and} \quad v_2 = w_2 - \frac{\langle w_2 , w_1 \rangle}{\|w_1\|^2} \cdot w_1.$$

Example

Example

Let $W = \{ (x, y, z) \in \mathbf{R}^3 : x - 2y + 2z = 0 \}$. Our usual techniques produce a basis $\gamma = \{ w_1, w_2 \} = \{ (2, 1, 0), (-2, 0, 1) \}$. If we want an orthogonal basis $\{ v_1, v_2 \}$, then we let $v_1 = w_1$ and

$$v_2 = w_2 - \frac{\langle w_2 \, , \, v_1 \rangle}{\| v_1 \|^2} v_1 = (-2, 0, 1) - \frac{-4}{5}(2, 1, 0) = (-\frac{2}{5}, \frac{4}{5}, 1)$$

Thus $\beta = \{ (2, 1, 0), (-2, 4, 5) \}$ is an orthogonal basis for W and $\beta' = \{ \frac{1}{\sqrt{5}}(2, 1, 0), \frac{1}{3\sqrt{5}}(-2, 4, 5) \}$ is an orthonormal basis.

Gram-Schmidt Orthogonalization

Theorem (Gram-Schmidt Orthogonalization Process)

Let V be an inner product space and $\gamma = \{ w_1, \dots, w_n \}$ a basis for a subspace W. If we define $\beta = \{ v_1, \dots, v_n \}$ by

$$v_k = \begin{cases} w_1 & \text{if } k = 1, \text{ and} \\ w_k - \sum_{j=1}^{k-1} \frac{\langle w_k \, , \, v_j \rangle}{\| v_j \|^2} v_j & \text{for } 2 \le k \le n, \end{cases}$$

then β is an orthogonal basis for W.

Return

Proof

Proof.

We proceed by induction on n. If $n = 1$ and $\gamma = \{ w_1 \}$, then let $v_1 = w_1$ and $\beta = \{ v_1 \}$ clearly works.

Now we assume that result if we are given $n - 1$ vectors in γ and consider a basis $\gamma = \{ w_1, \dots, w_n \}$ for W.

By the inductive hypothesis applied to $\gamma' = \{ w_1, \dots, w_{n-1} \}$ and $W' = \operatorname{Span}(\gamma')$, $\beta' = \{ v_1, \dots, v_{n-1} \}$ is an orthogonal basis for W'.

If $v_n = 0_V$, then $w_n \in \operatorname{Span}(\beta') = \operatorname{Span}(\gamma')$ which contradicts the linear independence of γ. (See the ▸ formula).

Since $\sum_{j=1}^{n-1} \frac{\langle w_n, v_j \rangle}{\| v_j \|^2} v_j \in \operatorname{Span}(\beta') = \operatorname{Span}(\gamma')$, v_n is a non-zero vector in $\operatorname{Span}(\gamma) = W$.

Proof

Proof Continued.

For $1 \leq k \leq n-1$,

$$\begin{aligned}\langle v_n \,,\, v_k \rangle &= \langle w_n \,,\, v_k \rangle - \sum_{j=1}^{n-1} \frac{\langle w_n \,,\, v_j \rangle}{\|v_j\|^2} \langle v_j \,,\, v_k \rangle \\ &= \langle w_n \,,\, v_k \rangle - \frac{\langle w_n \,,\, v_k \rangle}{\|v_k\|^2} \langle v_k \,,\, v_k \rangle \\ &= 0.\end{aligned}$$

Since $\beta' = \{ v_1, \dots, v_{n-1} \}$ is an orthogonal set of nonzero vectors, the above shows that $\beta = \{ v_1, \dots, v_n \}$ is also an orthogonal set of nonzero vectors. Hence β is linearly independent in W. Since $\dim(W) = n$, β must be an orthogonal basis as required. □

Example

Let $V = \mathrm{P}_2(\mathbf{R})$ and $\sigma = \{ 1, x, x^2 \}$ its standard basis. It is not hard to see that

$$\langle f \, , \, g \rangle = \int_0^1 f(t)g(t)\, dt$$

is an inner product on $\mathrm{P}_2(\mathbf{R})$. We want to find an orthogonal basis. We can use Gram-Schmidt. Thus in our formula, $\{ w_1, w_2, w_3 \} = \{ 1, x, x^2 \}$ We let $v_1 = w_1$. Then $\|v_1\|^2 = \langle v_1 \, , \, v_1 \rangle = 1$ and $\langle x \, , \, v_1 \rangle = \int_0^1 x\, dx = \frac{1}{2}$. Thus

$$v_2 = w_2 - \frac{\langle w_2 \, , \, v_1 \rangle}{\langle v_1 \, , \, v_1 \rangle} v_1 = x - \frac{1}{2}.$$

Example

Example (Continued)

Now $\|v_2\|^2 = \langle v_2 \, , \, v_2 \rangle = \int_0^1 (x - \frac{1}{2})^2 \, dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} x^2 \, dx = \frac{1}{12}$. Also $\langle w_3 \, , \, v_2 \rangle = \int_0^1 x^2 (x - \frac{1}{2}) \, dx = \frac{1}{12}$. Therefore

$$\begin{aligned} v_3 &= w_3 - \frac{\langle w_3 \, , \, v_2 \rangle}{\langle v_2 \, , \, v_2 \rangle} v_2 - \frac{\langle w_3 \, , \, v_1 \rangle}{\langle v_1 \, , \, v_1 \rangle} v_1 \\ &= x^2 - (1)(x - \frac{1}{2}) - \frac{1}{3} = x^2 - x + \frac{1}{6}. \end{aligned}$$

Therefore $\beta = \{\, 1, x - \frac{1}{2}, x^2 - x + \frac{1}{6} \,\}$ is an orthogonal basis for $P_2(\mathbf{R})$. To get an orthonormal basis, we need to normalize: $\{\, 1, 2\sqrt{3}(x - \frac{1}{2}), 6\sqrt{5}(x^2 - x + \frac{1}{6}) \,\}$ where I have hid the details to protect the innocent. In Example 5 in the text, the authors work the same example, but equip $P_2(\mathbf{R})$ with the inner product $\langle f \, , \, g \rangle = \int_{-1}^{1} f(x) g(x) \, dx$. This is therefore a different inner product space and hence the results are quite different.

An Important Corollary

Theorem

If V is a finite-dimensional inner product space with $\dim(V) = n$, then V has an orthonormal basis $\beta = \{ v_1, \dots, v_n \}$. If $v \in V$, then

$$v = \sum_{j=1}^{n} \langle v \, , v_j \rangle v_j . \qquad (\dagger)$$

Return

Proof.

We can let $\gamma = \{ w_1, \dots, w_n \}$ be any ordered basis for V. Then we can apply the Gram-Schmidt process to obtain an orthogonal basis $\beta' = \{ v_1', \dots, v_n' \}$. Then we can normalize to get β. The formula $(\dagger)$ follows immediately from $(\ddagger)$ in our earlier proposition. □

Fourier Coefficients

Definition

If β is an orthonormal subset of an inner product space V (possibly infinite dimensional), then the scalars $\langle v\ ,\ w\rangle$ where $w \in \beta$ are called the Fourier coefficients of v relative to β.

Remark

For us in Math 24, this is just a fancy name. In our standard setting where β is an orthonormal basis for V, they are just the magic scalars that appear in (†) on the previous slide.

The Matrix of a Linear Operator

Corollary

Let V be a finite-dimensional inner product space and let $\beta = \{ v_1, \dots, v_n \}$ be an orthonormal basis for V. If $T \in \mathcal{L}(V)$ is a linear operator, then $[T]_\beta = (A_{ij})$ where for all $1 \le i, j \le n$

$$A_{ij} = \langle T(v_j) \, , \, v_i \rangle. \qquad (*)$$

Proof.

We have $T(v_j) = \sum_{j=1}^{n} \langle T(v_j) \, , \, v_i \rangle v_i$. This means that $[T(v_j)]_\beta = (\langle T(v_j) \, , \, v_1 \rangle, \dots, \langle T(v_j) \, , \, v_n \rangle)$ and $A_{ij} = \langle T(v_j) \, , \, v_i \rangle$ as claimed. □

Remark

It is worth paying attention to the order of the indicies in the formula $(*)$. It will be a useful formula for us, but not if you remember it incorrectly.

Enough

- That is enough for today.