Math 24: Winter 2021
Lecture 24

Dana P. Williams

Dartmouth College

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Let's Get Started

1. We should be recording.
2. Remember, it is more comfortable for me if you turn on your video so that I feel like I am talking to real people.
3. The final will be administered in a manner similar to the prelim and midterm exams. Unless I am told that it breaks some college rule, my plan is to release it on the first day of the final period, Saturday, March 13, at 8am EST, and require it to be completed by Monday, March 15 at 10pm EST. There will be what I hope is a comfortable window to complete the exam.
4. The final will cover the entire course and I hope to get to parts of Section 6.6. We probably will not be covering all of sections 6.2 – 6.6. You will only be responsible for what we conver in lecture. More details later.
5. But first, are there any questions from last time?

Only So Much Room

Proposition

Suppose that V is an inner product space and $S = \{ v_1, \ldots, v_n \}$ is an orthogonal set of nonzero vectors. Then S is linearly independent.

Proposition

Suppose that $S = \{ v_1, \ldots, v_n \}$ is an orthogonal set of nonzero vectors in an inner product space V. If $v \in \operatorname{Span}(S)$, then

$$v = \sum_{k=1}^{n} \frac{\langle v\ ,\ v_k \rangle}{\|v_k\|^2} \cdot v_k.$$

In particular, if S is an orthonormal subset and $v \in \operatorname{Span}(S)$, then

$$v = \sum_{k=1}^{n} \langle v\ ,\ v_k \rangle \cdot v_k.$$

Orthogonal and Orthonormal Bases

Definition

If V is a finite-dimensional inner product space, then we call an ordered basis β an orthogonal basis if β is also an orthogonal set. If β is also orthonormal, we call β and orthonormal basis.

Example (Only The Best)

The standard ordered basis for $\mathbf{F}^n$ is an orthonormal basis.

Theorem (Gram-Schmidt Orthogonalization Process)

Let V be an inner product space and $\gamma = \{ w_1, \dots, w_n \}$ a basis for a subspace W. If we define $\beta = \{ v_1, \dots, v_n \}$ by

$$v_k = \begin{cases} w_1 & \text{if } k = 1, \text{ and} \\ w_k - \displaystyle\sum_{j=1}^{k-1} \frac{\langle w_k \, , \, v_j \rangle}{\|v_j\|^2} v_j & \text{for } 2 \le k \le n, \end{cases}$$

then β is an orthogonal basis for W.

An Important Corollary

Theorem

If V is a finite-dimensional inner product space with $\dim(V) = n$*, then V has an orthonormal basis $\beta = \{ v_1, \dots, v_n \}$. If $v \in V$, then*

$$v = \sum_{j=1}^{n} \langle v \,, v_j \rangle v_j.$$

Corollary

Let V be a finite-dimensional inner product space and let $\beta = \{ v_1, \dots, v_n \}$ be an orthonormal basis for V. If $T \in \mathcal{L}(V)$ is a linear operator, then $[T]_\beta = (A_{ij})$ where for all $1 \leq i, j \leq n$

$$A_{ij} = \langle T(v_j) \,, v_i \rangle.$$

Orthogonal Complements

Definition

Let S be a nonempty subset of an inner product space V. Then we let

$$S^{\perp} = \{\, v \in V : \langle v \,, w \rangle = 0 \text{ for all } w \in S \,\}.$$

We call $S^{\perp}$ the orthogonal complement of S (pronounced "S perp").

Remark (Low Hanging Fruit)

I leave it to you to prove that, for any nonempty subset S of V, $S^{\perp}$ is a subspace V. We also have $V^{\perp} = \{0_V\}$ and $\{0_V\}^{\perp} = V$.

Example

Let $v_0 = (a, b, c) \in \mathbf{R}^3$ and $S = \{v\}$. Then

$$S^\perp = \{\, v \in \mathbf{R}^3 : \langle v \;, v_0\rangle = 0 \,\}$$
$$= \{\, (x, y, z) \in \mathbf{R}^3 : ax + by + cz = 0 \,\}.$$

If you took Math 8, or otherwise have a proper education, you recognize this as the plane in $\mathbf{R}^3$ through the origin with normal vector v_0. Thus if $v_0 = e_3 = (0, 0, 1)$, then $S^\perp$ is just the xy-plane.

Subspaces

Theorem

Suppose that W is a finite-dimensional subspace of an inner product space V. Then V is the direct sum $W \oplus W^{\perp}$. In particular, given $v \in V$, there are unique vectors $w \in W$ and $z \in W^{\perp}$ such that $v = w + z$. If $\beta = \{ w_1, \dots, w_k \}$ is an orthonormal basis for W, then

$$w = \sum_{j=1}^{k} \langle v \,, w_j \rangle w_j. \tag{$\ddagger$}$$

Remark

Since $V = W \oplus W^{\perp}$, the map $P : V \to V$ sending $v \in V$ to $w \in W$ given by ($\ddagger$) is the linear operator we called the projection of V onto W along $W^{\perp}$. In the case, we call P the orthogonal projection of V onto W and w the orthogonal projection of v onto W.

Proof.

To see that $V = W \oplus W^\perp$ we need to verify that $W \cap W^\perp = \{0_V\}$ and that $W + W^\perp = V$.

Let w be as in (‡) and let $z = v - w$. I claim that $z \in W^\perp$. For this, it suffices (by a homework exercise) to see that $\langle z \,, w_j \rangle = 0$ for all $1 \leq j \leq k$. But

$$\begin{aligned}\langle z \,, w_j \rangle &= \Big\langle v - \sum_{i=1}^{k} \langle v \,, w_i \rangle w_i \,, w_j \Big\rangle \\ &= \langle v \,, w_j \rangle - \langle v \,, w_j \rangle = 0.\end{aligned}$$

Since we clearly have $w \in W$, we have $v = w + z \in W + W^\perp$. On the other hand, if $v \in W \cap W^\perp$, then $\langle v \,, v \rangle = 0$ and $v = \{0_V\}$. This completes the proof. $\square$

Corollary

Suppose that W is a finite-dimensional subspace of an inner product space V. If $v \in V$ and $w = P(v)$ is the orthogonal projection of v onto W, then w is the closest vector in W to v. That is, if $x \in W$, then

$$\|x - v\| \geq \|w - v\|.$$

Proof.

By the theorem, $v = w + z$ with $z \in W^\perp$. Then if $x \in W$, we have $w - x \in W$. Then the Pythagorean Equality (§6.1, #10) implies that

$$\begin{aligned} \|x - v\|^2 &= \|x - w - z\|^2 = \|z + (w - x)\|^2 \\ &= \|z\|^2 + \|w - x\|^2 \geq \|z\|^2 = \|w - v\|^2 \end{aligned}$$

The result follows by taking square roots. □

Break Time

Time for a break

Orthonormal Bases

Theorem

Let V be a n-dimensional inner product space. Suppose that $S = \{ v_1, \dots, v_k \}$ is an orthonormal set in V.

1. *Then S can be extended to an orthonormal basis $\beta = \{ v_1, \dots, v_k, v_{k+1}, \dots, v_n \}$ for V.*
2. *If $W = \mathrm{Span}(S)$, then $\{ v_{k+1}, \dots, v_k \}$ is a basis for $W^\perp$.*
3. *If W is any subspace of V the $\dim(V) = n = \dim(W) + \dim(W^\perp)$.*

A Lemma

Lemma

Suppose that V is a finite-dimension vector space and that W_1 and W_2 are subspaces. If $V = W_1 \oplus W_2$, then $\dim(V) = \dim(W_1) + \dim(W_2)$.

Proof of the Lemma.

We proved on the preliminary exam that, for any subspaces W_1 and W_2,

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2).$$

But if $V = W_1 \oplus W_2$ we have $W_1 + W_2 = V$ and $W_1 \cap W_2 = \{0_V\}$. So the result follows. $\square$

Proof.

(3) Item (3) follows from the lemma.

(1) We can extend S to a basis $\{ v_1, \dots, v_k, w_{k+1}, \dots, w_n \}$ by previous results. Then we can apply Gram-Schmidt to get an orthogonal basis $\{ v_1', \dots, v_n' \}$. A straightforward induction argument shows that $v_j' = v_j$ if $1 \leq j \leq k$. Now can normalize that last $n - k$ vectors to get an orthonormal basis.

(2) Clearly, $\{ v_{k+1}, \dots, v_n \}$ is an orthonormal subset of $W^\perp$. Hence it is linearly independent and contains $\dim(W^\perp) = n - k$ vectors. Hence it is a basis. $\square$

Break Time

Proposition

Suppose that V is a finite-dimensional inner product space over $\mathbf{F}$ and that $g : V \to \mathbf{F}$ is a linear map. Then there is a unique vector $z \in V$ such that $g(v) = \langle v , z \rangle$.

Proof.

Let $\beta = \{ v_1, \dots, v_n \}$ be an orthonormal basis for V. Let

$$z = \sum_{j=1}^{n} \overline{g(v_j)} v_j.$$

Define $h : V \to \mathbf{F}$ by $h(v) = \langle v , z \rangle$. Then if $1 \le k \le n$, we have

$$h(v_k) = \sum_{j=1}^{n} \langle v_k , \overline{g(v_j)} v_j \rangle = g(v_k).$$

Since β is a basis, this implies $h = g$.

Proof

Proof Continued.

To see that z is unique, suppose that we also have $g(v) = \langle v \, , z' \rangle$. Then $\langle v \, , z - z' \rangle = g(v) - g(v) = 0$ for all v. This forces $z = z'$ and proves uniqueness. $\square$

Example

Suppose that $g : \mathbf{R}^3 \to \mathbf{R}$ is linear. The previous proposition implies that there is a $v = (a, b, c) \in \mathbf{R}^3$ such that $g(x, y, z) = \langle (x \, , y, z), (a, b, c) \rangle = ax + by + cz$. This is just the map L_A where A is the 1×3-matrix $A = (a \; b \; c)$ which is just the matrix of g with respect to the standard bases.

The Adjoint

Theorem (The Adjoint)

Let V be a finite-dimensional inner product space and $T \in \mathcal{L}(V)$. Then there is a unique $T^* \in \mathcal{L}(V)$ such that

$$\langle T(v) , w \rangle = \langle v , T^*(w) \rangle \quad \textit{for all } v, w \in V.$$

The map T^* is called the adjoint of T.

Proof.

Fix $w \in V$ and define $g : V \to \mathbf{F}$ by $g(v) = \langle T(v) , w \rangle$. It is not hard to see that g is linear:

$$g(av + v') = \langle av + v' , w \rangle = a\langle v , w \rangle + \langle v' , w \rangle = ag(v) + g(v').$$

Hence there is a unique vector in V, which we call $T^*(w)$, such that $g(v) = \langle v , T^*(w) \rangle$. This defines a function $T^* : V \to V$.

Proof

Proof Continued.

I claim that T^* is linear. If $w, w' \in V$ and $a \in \mathbf{F}$, then for all $v \in V$, we have

$$\begin{aligned} \langle v \, , \, T^*(aw + w') \rangle &= \langle T(v) \, , \, aw + w' \rangle \\ &= \bar{a} \langle T(v) \, , \, w \rangle + \langle T(v) \, , \, w' \rangle \\ &= \bar{a} \langle v \, , \, T^*(w) \rangle + \langle v \, , \, T^*(w') \rangle \\ &= \langle v \, , \, aT^*(w) \rangle + \langle v \, , \, T^*(w') \rangle \\ &= \langle v \, , \, aT^*(w) + T^*(w') \rangle. \end{aligned}$$

Since this holds for all $v \in V$, we have $T^*(aw + w') = aT^*(w) = T^*(w')$ and $T^* \in \mathcal{L}(V)$ as claimed.

Now suppose $U \in \mathcal{L}(V)$ also satisfies $\langle T(v) \, , \, w \rangle = \langle v \, , \, U(w) \rangle$ for all $v, w \in V$. Then for each $w \in V$ and all $v \in V$, $\langle v \, , \, T^*(w) \rangle = \langle v \, , \, U(w) \rangle$. Thus $T^*(w) = U(w)$ for all w and $U = T^*$. □

Flipped

Remark

Note that

$$\langle w \,,\, T(v)\rangle = \overline{\langle T(v) \,,\, w\rangle} = \overline{\langle v \,,\, T^*(w)\rangle}$$
$$= \langle T^*(w) \,,\, v\rangle$$

Thus I have always though of the adjoint was saying that we can flip the operator T from one side of the inner product to the other provided we decorate it with a $*$ when we do.

Remark (Reading the Book)

If V is infinite dimensional and T is a linear operator on V, it turns out that T may not have an adjoint $T^* \in \mathcal{L}(V)$ that satisfies $\langle T(v) \,,\, w\rangle = \langle v \,,\, T^*(w)\rangle$ for all $v, w \in V$. But if it does, it is still unique.

Matrix Adjoints

Theorem

Let V be a finite-dimensional inner product space with an orthonormal basis β. If $T \in \mathcal{L}(V)$, then

$$[T^*]_\beta = [T]_\beta^*.$$

Remark

This result gives us a concrete way to compute what T^* is! However, keep in mind that this works for any *orthonormal* basis. It is usually not correct for an ordered basis which is not orthonormal.

Proof

Proof Continued.

To make the notation more readable, let $A = [T]_\beta$ and $B = [T^*]_\beta$. Let $\beta = \{ v_1, \dots, v_n \}$. Then

$$\begin{aligned} B_{ij} &= \langle T^*(v_j)\ ,\ v_i \rangle = \langle v_j\ ,\ T(v_i) \rangle \\ &= \overline{\langle T(v_i)\ ,\ v_j \rangle} = \overline{A_{ji}}. \end{aligned}$$

That is $B = A^*$. □

Remark

For a given orthonormal basis β, it is perfectly possible that $[T]_\beta^* = [T]_\beta$. Then $T = T^*$. We call such operators self adjoint.

Matrix Version

Corollary

If $A \in M_{n\times n}(\mathbf{F})$, then $(L_A)^* = L_{A^*}$.

Proof.

Let σ be the standard basis for $\mathbf{F}^n$. Then σ is also an orthonormal basis and $[L_A]_\sigma = A$. Then by the theorem, $[L_A^*]_\sigma = A^*$ and $L_A^* = L_{A^*}$. $\square$

Properties of the Adjoint

Theorem

Suppose that V is an inner product space and $T, U \in \mathcal{L}(V)$. If V is not finite-dimensional, assume that both T and U have adjoints.

1. $(cT + U)^* = \overline{c}T^* + U^*$.
2. $(TU)^* = U^*T^*$.
3. $(T^*)^* = T^{**} = T$.
4. $I_V^* = I_V$.

Remark

To be pedantic, part of the assertion in the above result is that in the infinite-dimensional case, $cT + U$, TU, and I_V always have adjoints if T and U do. Of course, this is automatic in the finite-dimensional case.

Proof

Proof.

These results are all proved similarly. For example, to prove item (2) consider

$$\begin{aligned}\langle TU(v)\,,\,w\rangle &= \langle T(U(v))\,,\,w\rangle = \langle U(v)\,,\,T^*(w)\rangle \\ &= \langle v\,,\,U^*(T^*(w))\rangle = \langle v\,,\,U^*T^*(w)\rangle.\end{aligned}$$

By uniqueness, $(TU)^* = U^*T^*$. $\square$

Corollary

Suppose $A, B \in M_{n\times n}(\mathbf{F})$ and $c \in \mathbf{F}$.

1. $(cA + B)^* = \overline{c}A^* + B^*$.
2. $(AB)^* = B^*A^*$.
3. $A^{**} = A$.
4. $I_n^* = I_n$.

Proof.

I just prove item (2). Then

$$L_{(AB)^*} = (L_{AB})^* = (L_A L_B)^* = L_B^* L_A^* = L_{B^*} L_{A^*} = L_{B^*A^*}.$$

Therefore $(AB)^* = B^*A^*$. $\square$

Enough

1. That is enough for today.