Math 24: Winter 2021
Lecture 25

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Let's Get Started

1. We should be recording.
2. The final will be administered in a manner similar to the prelim and midterm exams and will available for download Saturday March 13 at 8am until Monday March 15 at 10pm.
3. The final will be cumulative. In particular, it will cover whatever parts of Chapter 6 we manage to cover by Wednesday. You will only be responsible for what we cover in lecture.
4. Zinn's Law and the Law of Exponential Interest.
5. But first, are there any questions from last time?

Review

Definition

Let S be a nonempty subset of an inner product space V. Then we let

$$S^{\perp} = \{ v \in V : \langle v , w \rangle = 0 \text{ for all } w \in S \}.$$

We call $S^{\perp}$ the orthogonal complement of S (pronounced "S perp").

Theorem

Suppose that W is a finite-dimensional subspace of an inner product space V. Then V is the direct sum $W \oplus W^{\perp}$. In particular, given $v \in V$, there are unique vectors $w \in W$ and $z \in W^{\perp}$ such that $v = w + z$. If $\beta = \{ w_1, \dots, w_k \}$ is an orthonormal basis for W, then

$$w = \sum_{j=1}^{k} \langle v , w_j \rangle w_j. \tag{‡}$$

Review

Remark

Since $V = W \oplus W^\perp$, the map $P : V \to V$ sending $v \in V$ to $w \in W$ given by (1) is the linear operator we called the projection of V onto W along $W^\perp$. In the case, we call P the orthogonal projection of V onto W and w the orthogonal projection of v onto W.

Theorem (The Adjoint)

Let V be a finite-dimensional inner product space and $T \in \mathcal{L}(V)$. Then there is a unique $T^* \in \mathcal{L}(V)$ such that

$$\langle T(v)\, ,\, w\rangle = \langle v\, ,\, T^*(w)\rangle \quad \text{for all } v, w \in V.$$

The map T^* (pronounced "T star") is called the adjoint of T.

Matrix Adjoints

Theorem

Let V be a finite-dimensional inner product space with an orthonormal basis β. If $T \in \mathcal{L}(V)$, then

$$[T^*]_\beta = [T]_\beta^*.$$

Theorem

Suppose that V is an inner product space and $T, U \in \mathcal{L}(V)$. If V is not finite-dimensional, assume that both T and U have adjoints.

1. $(cT + U)^* = \overline{c}T^* + U^*$.
2. $(TU)^* = U^*T^*$.
3. $(T^*)^* = T^{**} = T$.
4. $I_V^* = I_V$.

Goals Upgraded

Remark

Since we like to diagonalize operators on ordinary finite-dimensional vector spaces V by finding a basis of eigenvectors for V, we can upgrade our quest to try to find an orthonormal basis of eigenvectors for a linear operator on a finite-dimensional inner product space V. We say that such an operator is orthogonally diagonalizable.

Remark

Note that if $T \in \mathcal{L}(V)$ is orthogonally diagonalizable then there is an orthonormal basis β such that $[T]_\beta$ is diagonal. But then so is $[T^*]_\beta = [T]_\beta^*$. But then

$$[T^*T]_\beta = [T^*]_\beta[T]_\beta = [T]_\beta[T^*]_\beta = [TT^*]_\beta.$$

This means that T and T^* commute; that is, $T^*T = TT^*$. What else can we add if V is a real vector space?

Normal Operators

Definition

A linear operator $T \in \mathcal{L}(V)$ on an inner product space V is called normal if $T^*T = TT^*$. Similarly, $A \in M_{n\times n}(\mathbf{F})$ is called normal of $A^*A = AA^*$.

Example

Let $T_\theta : \mathbf{R}^2 \to \mathbf{R}^2$ be rotation by θ radians counterclockwise around the origin. Then with respect to the standard basis,

$$[T_\theta]_\sigma = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}. \tag{1}$$

Then you could check that $[T_\theta]_\sigma^*[T_\theta]_\sigma = I_2 = [T_\theta]_\sigma[T_\theta]_\sigma^*$ which implies that T_θ is normal. Alternatively, note that $T_\theta^* = T_{-\theta}$ and see directly that $T_\theta^* T_\theta = I_{\mathbf{R}^2} = T_\theta T_\theta^*$.

Bummer

Remark

If $0 < \theta < \pi$, then we have already seen that $T_\theta; \mathbf{R}^2 \to \mathbf{R}^2$ has no eigenvectors. So normality of $T \in \mathcal{L}(V)$ is not generally going to imply that T is orthogonally diagonalizable for a real vector space V. Nevertheless, it will suffice in a complex vector space. Proving this will be our next goal.

Normal Operators

Theorem

Let V be a real or complex inner product space and let $T \in \mathcal{L}(V)$ be normal.

1. *For all $v \in V$, $\|T(v)\| = \|T^*(v)\|$.*
2. *For all $c \in \mathbf{F}$, $T - cI_V$ is normal.*
3. *If v is an eigenvector for T with eigenvalue λ, then v is also an eigenvector for T^* with eigenvalue $\overline{\lambda}$.*
4. *If v_1 and v_2 are eigenvectors for T with distinct eigenvalues, then v_1 and v_2 are orthogonal.*

Proof.

(1) We have

$$\begin{aligned} \|T(v)\|^2 &= \langle T(v) , T(v) \rangle = \langle T^*T(v) , v \rangle \\ &= \langle TT^*(v) , v \rangle = \langle T^*(v) , T^*(v) \rangle = \|T^*(v)\|^2. \end{aligned}$$

This proves (1), and the proof of (2) is routine (note that $(T - cI_V)^* = T^* - \overline{c}I_V$ and compute).

Proof

Proof Continued.

(3) Suppose $T(v) = \lambda v$ with $v \neq 0_V$. Then $U = T - \lambda I_V$ is normal with $U^* = T^* - \overline{\lambda} I_V$. Furthermore, $\|U(v)\| = 0$. By part (1), $\|U^*(v)\| = 0$ and v is an eigenvector for T^* with eigenvalue $\overline{\lambda}$ as claimed.

(4) Suppose $T(v_1) = \lambda_1 v_1$ and $T(v_2) = \lambda_2 v_2$ with $\lambda_1 \neq \lambda_2$. Then

$$\begin{aligned} \lambda_1 \langle v_1 \, , \, v_2 \rangle &= \langle \lambda_1 v_1 \, , \, v_2 \rangle = \langle T(v_1) \, , \, v_2 \rangle \\ &= \langle v_1 \, , \, T^*(v_2) \rangle = \langle v_1 \, , \, \overline{\lambda}_2 v_2 \rangle = \lambda_2 \langle v_1 \, , \, v_2 \rangle. \end{aligned}$$

Therefore $(\lambda_2 - \lambda_1)\langle v_1 \, , \, v_2 \rangle = 0$. Since $\lambda_2 - \lambda_1 \neq 0$, we must have $\langle v_1 \, , \, v_2 \rangle = 0$. □

Break Time

Time for a break and some questions.

Orthogonal Complements Again

Lemma

Suppose that V is an inner product space and that $T \in \mathcal{L}(V)$. If W is T-invariant, then $W^\perp$ is T^*-invariant.

Proof.

Suppose $w \in W$ and $z \in W^\perp$. Then $T(w) \in W$ and

$$\langle T^*(z) \, , \, w \rangle = \langle z \, , \, T(w) \rangle = 0.$$

Since this holds for all $w \in W$, it follows that $T^*(z) \in W^\perp$. But z was an arbitrary element of $W^\perp$. □

Key Lemma

Lemma

Suppose that V is an inner product space and that $T \in \mathcal{L}(V)$ is normal. Suppose that W is a subspace that is both T-invariant and T^*-invariant. Then $W^\perp$ is T-invariant and the restriction $T_{W^\perp}$ is normal in $\mathcal{L}(W^\perp)$.

Proof.

Since W is T^*-invariant and $T^{**} = T$, $W^\perp$ is T-invariant. By symmetry, is also T^* invariant. Hence we can form the operators $T_{W^\perp}$ and $T^*_{W^\perp}$ on the inner product space $W^\perp$. If $x, y \in W^\perp$, then

$$\langle T_{W^\perp}(x) \, , \, y\rangle = \langle T(x) \, , \, y\rangle = \langle x \, , \, T^*(y)\rangle = \langle x \, , \, T^*_{W^\perp}(y)\rangle.$$

Since the adjoint is unique, this means that $T^*_{W^\perp} = (T_{W^\perp})^*$ as operators on $W^\perp$. But $T_{W^\perp}$ and $T^*_{W^\perp}$ commute because T and T^* do. Hence $T_{W^\perp}$ is normal as claimed. □

The Main Result

Theorem

Suppose that T is a linear operator on a finite-dimensional inner product space V and that the characteristic polynomial of T splits. If T is normal, then T is orthogonally diagonalizable.

Proof.

We will work by induction on $n = \dim(V)$. If $n = 1$, then any unit vector $v \in V$ is an eigenvector and $\beta = \{v\}$ is an orthonormal basis. So we assume that the result holds for inner product spaces of dimension $n - 1$ for for some $n \geq 2$. We consider V with $\dim(V) = n$. Since the characteristic polynomial of T splits, it must have at least one root and hence T has an eigenvector v_1. Since T is normal, v_1 is also an eigenvector for T^*.

Proof

Proof Continued.

We can assume $\|v_1\| = 1$ and let $W = \text{Span}(\{v_1\})$. Then W is both T-invariant and T^*-invariant. By our lemma, $W^\perp$ is T-invariant and $T_{W^\perp}$ is normal. Furthermore, its characteriestic polynomial divides that of T and must split as well. Since $\dim(W^\perp) = n - 1$, the induction hypotheses implies that $W^\perp$ has an orthonormal basis of eigenvectors $\{ v_2, \dots, v_n \}$ for $T_{W^\perp}$. Since $T_{W^\perp}$ is the restriction of T, $\{ v_2, \dots, v_n \}$ is also an orthonormal set of eigenvectors for T. But then $\beta = \{ v_1, \dots, v_n \}$ is an orthonormal set of eigenvectors for T. In particular, β is linearly independent. Since $\dim(V) = n$, β is a basis and we are done. □

Complex Case

Corollary

Suppose that T is a linear operator on a finite-dimensional complex inner product space. Then T is orthogonally diagonalizable if and only if T is normal.

Proof.

If T is orthogonally diagonalizable, we have already pointed out that T must be normal.

Since we are working over $\mathbf{C}$, the characteristic polynomial of T always splits. Therefore if T is normal, it is orthogonally diagonalizable by the previous theorem. □

Break Time

Time for a well earned break and a few questions.

Self-Adjoint Operators

Definition

An operator T on an inner product space V is called self-adjoint if $T = T^*$. Similarly, a matrix $A \in M_{n\times n}(\mathbf{F})$ is called self-adjoint if $A = A^*$. The textbook sometimes uses Hermitian in place of self-adjoint.

Remark

An operator T is self-adjoint if and only if $[T]_\beta$ is self-adjoint for some and hence all orthonormal bases β. Furthermore, a real matrix is self-adjoint if and only if it is symmetric.

Cool Result

Lemma

Let T be a self-adjoint operator on a finite-dimensional inner product space V.

1. *All the eigenvalues of T are real.*
2. *The characteristic polynomial of T splits.*

Remark

Item (1) only has content if V is a complex vector space, while item (2) only has content if V is a real vector space.

Proof

Proof.

(1) Let v be an eigenvector for T with eigenvalue λ. We can assume that $\|v\| = 1$. Then

$$\begin{aligned} \lambda = \lambda\langle v\,,\, v\rangle = \langle \lambda v\,,\, v\rangle = \langle T(v)\,,\, v\rangle \\ = \langle v\,,\, T(v)\rangle = \langle v\,,\, \lambda v\rangle = \overline{\lambda}\langle v\,,\, v\rangle = \overline{\lambda}. \end{aligned}$$

Therefore $\lambda = \overline{\lambda}$ and λ is real.

(2) As per our remark, we can assume that V is a real vector space with $n = \dim(V)$. Let β be an orthonormal basis for V and let $A = [T]_\beta$. Then A is self-adjoint (which simply means symmetric since V is a real vector space). Let T_A be the left-multiplication operator on $\mathbf{C}^n$ given by $T_A(x) = Ax$ for $x \in \mathbf{C}^n$.

Proof

Proof Continued.

If σ is the standard (orthonormal) basis for $\mathbf{C}^n$, then $[T_A]_\sigma = A$. Hence T_A is self-adjoint. By part (1), the eigenvalues of T_A are all real. Since we are working over $\mathbf{C}$, the characteristic polynomial of T_A splits into factors of the form $(t - \lambda)$ with each $\lambda \in \mathbf{R}$. Hence the characteristic polynomial of T_A splits over $\mathbf{R}$. But the characteristic polynomial of T_A is the same as that for A which is the same as that for T. Hence the characteristic polynomial of T splits over $\mathbf{R}$. □

The Big Corollary

Corollary

Suppose that T is a self-adjoint operator on a finite-dimensional real inner product space. Then T is self-adjoint if and only if T is orthogonally diagonalizable.

Proof.

If T is self-adjoint, then by our lemma, its characteristic polynomial splits. Since self-adjoint operators are obviously normal, T is orthogonally diagonalizable.

I will leave the converse as an exercise. □

Enough

1. That is enough for today.