Math 24: Winter 2021
Lecture 26

Dana P. Williams

Dartmouth College

Monday, March 8, 2021

Let's Get Started

1. We should be recording.
2. The final will be administered in a manner similar to the prelim and midterm exams and will available for download Saturday March 13 at 8am until Monday March 15 at 10pm.
3. The final will be cumulative. In particular, it will cover whatever parts of Chapter 6 we manage to cover by Wednesday. You will only be responsible for what we cover in lecture.
4. Zinn's Law and the Law of Exponential Interest.
5. But first, are there any questions from last time?

Review

Definition

A linear operator $T \in \mathcal{L}(V)$ on an inner product space V is called normal if $T^*T = TT^*$. Similarly, $A \in M_{n\times n}(\mathbf{F})$ is called normal of $A^*A = AA^*$.

Theorem

Let V be a real or complex inner product space and let $T \in \mathcal{L}(V)$ be normal.

1. *For all $v \in V$, $\|T(v)\| = \|T^*(v)\|$.*
2. *For all $c \in \mathbf{F}$, $T - cI_V$ is normal.*
3. *If v is an eigenvector for T with eigenvalue λ, then v is also an eigenvector for T^* with eigenvalue $\overline{\lambda}$.*
4. *If v_1 and v_2 are eigenvectors for T with distinct eigenvalues, then v_1 and v_2 are orthogonal.*

Review

Theorem

Suppose that T is a linear operator on a finite-dimensional inner product space V whose characteristic polynomial splits. If T is normal, then T is orthogonally diagonalizable.

Corollary

Suppose that T is a linear operator on a finite-dimensional complex inner product space. Then T is orthogonally diagonalizable if and only if T is normal.

Corollary

Suppose that T is a self-adjoint operator on a finite-dimensional real inner product space. Then T is self-adjoint if and only if T is orthogonally diagonalizable.

Unitary and Orthogonal Operators

Definition

Suppose that V is an inner product space over $\mathbf{F}$ and $T \in \mathcal{L}(V)$. Then T is called isometric if $\|T(v)\| = \|v\|$ for all $v \in V$. If $\mathbf{F} = \mathbf{R}$, the an isometric operator is called an orthogonal operator and if $\mathbf{F} = \mathbf{C}$ an isometric operator is called a unitary operator.

Example

Let $T_\theta : \mathbf{R}^2 \to \mathbf{R}^2$ the rotation operator whose matric with respect to the standard basis is $A_\theta = \left(\begin{smallmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{smallmatrix}\right)$. It is clear—either geometrically or by a short calculation—that each T_θ is isometric. Hence T_θ is an orthogonal operator.

Lemma

Let V be a finite-dimensional vector space. Suppose that $T, S \in \mathcal{L}(V)$ and $TS = I_V$. Then both T and S are invertible with $T^{-1} = S$ and $S^{-1} = T$.

Proof.

Let β be an ordered basis for V. Then $[TS]_\beta = [T]_\beta [S]_\beta$. Therefore $[T]_\beta$ is invertible and $[T]_\beta^{-1} = [S]_\beta$. This implies T is invertible and that $[T^{-1}]_\beta = [T]_\beta^{-1} = [S]_\beta$. Hence $T^{-1} = S$. The proof for S is similar. □

A Lemma

Lemma

Suppose that U is a self-adjoint operator on an inner product space V. Suppose also that

$$\langle x\,,\,U(x)\rangle = 0 \quad \textit{for all } x \in V.$$

Then $U = T_0$. (Recall that T_0 is the zero operator on V.)

Proof.

Since T is self-adjoint, it is orthogonally diagonalizable and there is an (orthonormal) basis $\beta = \{ v_1, \dots, v_n \}$ of eigenvectors for T with eigenvalues $\lambda_1, \dots, \lambda_n$, respectively. Then for $1 \leq k \leq n$, we have

$$\lambda_k = \lambda_k \langle v_k\,,\,v_k\rangle = \langle \lambda_k v_k\,,\,v_k\rangle = \langle T(v_k)\,,\,v_k\rangle = 0.$$

Therefore $\lambda_k = 0$ and $T(v_k) = 0$ for all $v_k \in \beta$. Since β is a basis, this implies $T = T_0$ as claimed. □

Isometric Operators

Theorem

Let $T \in \mathcal{L}(V)$ for a finite-dimensional inner product space V. Then the following are equivalent.

1. *T is isometric.*
2. *T is invertible with $T^{-1} = T^*$.*
3. *$TT^* = I_V$.*
4. *$T^*T = I_V$.*
5. *T preserves inner products in that $\langle T(x)\, ,\, T(y)\rangle = \langle x\, ,\, y\rangle$ for all $x, y \in V$.*
6. *If β is an orthonormal basis for V, then $T(\beta)$ is an orthonormal basis for V.*
7. *The is an orthonormal basis for V such that $T(\beta)$ is an orthonormal basis for V.*

Proof.

We have (2)$\iff$(3)$\iff$(4) since V is finite dimensional.

(1)$\Longrightarrow$(4): For all $x \in V$,

$$\langle x\,,\,x\rangle = \|x\|^2 = \|T(x)\|^2 = \langle T(x)\,,\,T(x)\rangle = \langle x\,,\,T^*T(x)\rangle.$$

Therefore $\langle x\,,\,(I_V - T^*T)(x)\rangle = 0$ for all $x \in V$. Since $U = I_V - T^*T)$ is self-adjoint by our lemma, $T^*T = I_V$. Hence T is invertible and $T^{-1} = T^*$.

(4)$\Longrightarrow$(5): We have $\langle x\,,\,y\rangle = \langle T^*T(x)\,,\,y\rangle = \langle T(x)\,,\,T(y)\rangle$.

(5)$\Longrightarrow$(6): Let $\beta = \{\, v_1, \dots, v_n \,\}$ be an orthonormal basis for V. Then $T(\beta) = \{\, T(v_1), \dots, T(v_n) \,\}$ and $\langle T(v_i)\,,\,T(v_j)\rangle = \langle v_i\,,\,v_j\rangle = \delta_{ij}$. This shows that $T(\beta)$ is an orthonormal basis as well.

(6)$\Longrightarrow$(7): This is immediate.

Proof

Proof Continued.

(7)$\Longrightarrow$(1): Suppose that $\beta = \{ v_1, \dots, v_n \}$ is an orthonormal basis and that $v \in V$. Then $v = \sum_{j=1}^{n} a_j v_j$ (where $a_j = \langle v \, , \, v_j \rangle$). Then

$$\|v\|^2 = \langle \sum_{j=1}^{n} a_j v_j \, , \, \sum_{k=1}^{n} a_k v_k \rangle = \sum_{j=1}^{n} \sum_{k=1}^{n} a_j \overline{a_k} \langle v_j \, , \, v_k \rangle = \sum_{j=1}^{n} |a_j|^2.$$

But $T(v) = \sum_{j=1}^{n} a_j T(v_j)$. Thus if $T(\beta)$ is also an orthonormal basis, then same computation shows that

$$\|T(v)\|^2 = \sum_{j=1}^{n} |a_j|^2 = \|v\|^2.$$

Thus T is isometric. □

Break Time

Time for a brake and some questions.

Matrices

Definition

We say that $A \in M_{n\times n}(\mathbf{R})$ is orthogonal if $A^t A = I_n = AA^t$. We say that $A \in M_{n\times n}(\mathbf{C})$ is unitary if $A^* A = I_n = AA^*$.

Remark

If A is an orthogonal real $n \times n$-matrix then it is unitary when viewed as an element of $M_{n\times n}(\mathbf{C})$.

Unitary Matrices

Proposition

Let $A \in M_{n\times n}(\mathbf{C})$. Then the following are equivalent.

1. *A is unitary.*
2. $A^* = A^{-1}$.
3. $A^*A = I_n$.
4. $AA^* = I_n$.
5. *A^* is unitary.*
6. *The rows of A are an orthonormal basis for $\mathbf{C}^n$*
7. *The columns of A are an orthonormal basis for $\mathbf{C}^n$.*

Remark

I leave it to you to sort our the corresponding statement for orthogonal real matrices.

Proof

Proof.

The equivalence of (1) and (2) is essentially by definition. The equivalence of (2), (3), and (4) is general matrix inverse nonsense. Since $A^{**} = A$, it is clear that (5) is equivalent to (1)–(4).

Let v_j be the j^{th}-column of A. Then

$$(A^*A)_{ij} = \sum_{k=1}^{n} (A^*)_{ik} A_{kj} = \sum_{k=1}^{n} \overline{A_{ki}} A_{kj} = \langle v_j \, , \, v_i \rangle$$

Thus $A^*A = I_n$ if and only if $\langle v_j \, , \, v_i \rangle = \delta_{ij}$. That is, (6)$\Longleftrightarrow$(3). A similar argument shows that (7)$\Longleftrightarrow$(4). □

Orthgonal Diagonalization

Remark

Suppose that A is either a complex normal $n \times n$-matrix [or a symmetric real $n \times n$-matrix]. Then our results from Friday's lecture applied to L_A imply that there is an orthonormal basis $\beta = \{ v_1, \dots, v_n \}$ of eigenvectors for A. If we let $U = [v_1 \ \cdots \ v_n$ be the matrix whose columns are the eigenvectors from β, then we have $D = Q^{-1}AQ$ where D is diagonal. Since β is orthonormal, Q is unitary [orthogonal]. Thus

$$D = Q^* AQ$$

and we say that A is unitarily equivalent [orthogonally equivalent] to a diagonal matrix. In general, we say that two matrices are unitarily equivalent [orthogonally equivalent] if there is a unitary [orthogonal] matrix P such that $A = P^* BP$.

An Example

Example

Let $A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$. Since A is symmetric, we ought to be able to orthogonally diagonalize A. Suppose that someone nice gives us the characteristic polynomial $p(\lambda) = -(\lambda - 4)(\lambda - 1)^2$. The easy bit is noticing that

$$A - 4I_3 = \begin{pmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \\ 1 & -1 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \text{ so that}$$

$v_3 = (1, -1, 1)$ is an eigenvector with eigenvalue 4.

Example

Example (Continued)

Things get more interesting when we consider
$A - I_3 = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ giving us a basis $\{ w_1, w_2 \} = \{ (1,1,0), (-1,0,1) \}$ for the E_1-eigenspace. As predicted by the theory, $\langle v_3 \, , w_k \rangle = 0$ for $k = 1, 2$ and $v_3 \in E_1^\perp$. So now we apply Gram-Schmidt to $\{ w_1, w_2 \}$. We let $v_1 = w_1$ and

$$v_2 = w_2 - \frac{\langle w_2 \, , v_1 \rangle}{\langle v_1 \, , v_1 \rangle} v_1 = (-1,0,1) - \frac{-1}{2}(1,1,0)$$
$$= (-\frac{1}{2}, \frac{1}{2}, 1).$$

Thus $\beta' = \{ (1,1,0), (-1,1,2), (1,-1,1) \}$ is an orthogonal basis of eigenvectors.

Example

Example (Continued)

Now we get an orthogonal matrix $Q = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix}$ such that $D = Q^tAQ$ where $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$.

Example

Now let's orthogonally diagonalize $A = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$. Here the characteristic polynomial is $p(\lambda) = \lambda^2 - 1$. Thus the eigenvalues are ± 1. Since $A - I_2 = \begin{pmatrix} -1 & i \\ -i & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & -i \\ 0 & 0 \end{pmatrix}$, $v_1 = (i, 1) \in E_1$. Similarly, we get $(-i, 1) \in E_{-1}$. Then $Q = \frac{1}{\sqrt{2}}\begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}$ is a unitary matrix and

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = Q^*AQ = \begin{pmatrix} \frac{-i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \begin{pmatrix} \frac{i}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

Break Time

Time for a break and some questions.

Rigid Motions

Definition

Let V be a real inner product space. A function $f : V \to V$ is called a rigid motion if it is distance preserving in that

$$\|f(x) - f(y)\| = \|x - y\| \quad \text{for all } x, y \in V. \qquad (*)$$

Remark

Note that we are not making any assumptions on f except for $(*)$. In particular, we are not asserting that f is linear. If f were linear, then f would simply be an orthogonal transformation. A translation—that is a map of the form $f(x) = x + v_0$ for some $v_0 \in V$—is also a rigid motion. It is not hard to see that that composition of rigid motions is a rigid motion.

All Rigid Motions

Theorem

Suppose that $f : V \to V$ *is a rigid motion of a real inner product space* V*. Then there exists a unique orthogonal transformation* $T \in \mathcal{L}(V)$ *and a translation* g *such that* $f = g \circ T$*. That is,* $f(x) = T(x) + v_0$ *for a unique orthogonal transformation* T *and unique vector* $v_0 \in V$*.*

Remark

Note that we simply get an orthogonal transformation if $v_0 = 0_V$ and a translation if $T = I_V$.

Proof

Proof.

Let $v_0 = f(0_V)$ and define $T : V \to V$ by $T(x) = f(x) - f(0) = f(x) - v_0$. Since T is the composition of f with translation by $-v_0$, T is a rigid motion.

I claim that T is inner product preserving. Note that

$$\|T(x)\| = \|f(x) - f(0_V)\| = \|x - 0_V\| = \|x\|.$$

Thus T is isometric. Furthermore,

$$\begin{aligned}\|T(x) - T(y)\|^2 &= \langle T(x) - T(y)\,,\, T(x) - T(y)\rangle \\ &= \|T(x)\|^2 - \langle T(x)\,,\, T(y)\rangle - \langle T(y)\,,\, T(x)\rangle + \|T(y)\|^2 \\ &= \|x\|^2 - 2\langle T(x)\,,\, T(y)\rangle + \|y\|^2.\end{aligned}$$

Proof

Proof Continued.

Since T is a rigid motion, we also have

$$\|T(x) - T(y)\|^2 = \|x - y\|^2 = \|x\|^2 - 2\langle x , y\rangle + \|y\|^2.$$

This means

$$\langle T(x) , T(y)\rangle = \langle x , y\rangle \quad \text{for all } x, y \in V.$$

This proves the claim.

Now I claim that T is linear. □

Proof

Proof Continued.

Consider

$$\begin{aligned}
&\|T(x+ay) - T(x) - aT(y)\|^2 \\
&\qquad = \|\underbrace{[T(x+ay) - T(x)]}_{v} - \underbrace{aT(y)}_{w}\|^2 \\
&\qquad = \|v\|^2 - 2\langle v\,,\,w\rangle + \|w\|^2 \\
&\qquad = \|T(x+ay) - T(x)\|^2 + a^2\|T(y)\|^2 \\
&\qquad\qquad\qquad - 2a\langle T(x+ay) - T(x)\,,\,T(y)\rangle \\
&\qquad = \|(x+ay) - x\|^2 + a^2\|y\|^2 \\
&\qquad\quad - 2a\big[\langle T(x+ay)\,,\,T(y)\rangle - \langle T(x)\,,\,T(y)\rangle\big] \\
&\qquad = 2a^2\|y\|^2 - 2a\big[\langle x+ay\,,\,y\rangle - \langle x\,,\,y\rangle\big] \\
&\qquad = 2a^2\|y\|^2 - 2a\big[\langle x\,,\,y\rangle + a\langle y\,,\,y\rangle - \langle x\,,\,y\rangle\big] = 0
\end{aligned}$$

Therefore $T(x+ay) = T(x) + aT(y)$ and T is linear as claimed.

Proof

Proof Continued.

Since we already saw that T preserved inner products, it follows that T is orthgonal and we have written $f = g \circ T$ for an orthogonal operator T and a translation $g(x) = x + v_0$.

To establish uniqueness, suppose $f(x) = U(x) + u_0$ with U orthogonal. Since $v_0 = f(0_V) = U(0_V) + u_0 = u_0$, this implies $T(x) = U(x)$ for all $x \in V$. That is, $U = T$ and we are done. □

Enough

1. That is enough for today.