Math 24: Winter 2021
Lecture 27

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Let's Get Started

1. We should be recording.
2. The final will be administered in a manner similar to the prelim and midterm exams and will available for download Saturday March 13 at 8am until Monday March 15 at 10pm.
3. You will have 6 hours to work the final.
4. I will have office hours Thursday and Friday 10-11.
5. But first, are there any questions from last time?

Review

Theorem

Let $T \in \mathcal{L}(V)$ for a finite-dimensional inner product space V. Then the following are equivalent.

1. *T is isometric.*
2. *T is invertible with $T^{-1} = T^*$.*
3. *$TT^* = I_V$.*
4. *$T^*T = I_V$.*
5. *T preserves inner products in that $\langle T(x), T(y)\rangle = \langle x, y\rangle$ for all $x, y \in V$.*
6. *If β is an orthonormal basis for V, then $T(\beta)$ is an orthonormal basis for V.*
7. *There is an orthonormal basis β for V such that $T(\beta)$ is an orthonormal basis for V.*

Review

Definition

We say that $A \in M_{n\times n}(\mathbf{R})$ is orthogonal if $A^tA = I_n = AA^t$. We say that $A \in M_{n\times n}(\mathbf{C})$ is unitary if $A^*A = I_n = AA^*$.

Remark

If A is an orthogonal real $n \times n$-matrix then it is unitary when viewed as an element of $M_{n\times n}(\mathbf{C})$.

Review

Proposition

Let $A \in M_{n \times n}(\mathbf{C})$. Then the following are equivalent.

1. *A is unitary.*
2. $A^* = A^{-1}$.
3. $A^*A = I_n$.
4. $AA^* = I_n$.
5. *A^* is unitary.*
6. *The rows of A are an orthonormal basis for $\mathbf{C}^n$*
7. *The columns of A are an orthonormal basis for $\mathbf{C}^n$.*

Remark

I left the proof of this and the corresponding statement for orthogonal matrices to you. Is so doing, you will have observed that A is unitary or orthogonal exactly when L_A is isometric (aka orthogonal or unitary).

A Little More Business

Lemma

Let T be an orthogonal operator on $\mathbf{R}^2$. Then $\det(T) = \pm 1$. If $\det(T) = 1$ then T is a rotation.

Proof.

Let σ be the standard ordered basis in $\mathbf{R}^2$ and $A = [T]_\sigma$. Then $A^t = [T^*]$ and since T is orthogonal, $T^*T = I_{\mathbf{R}^2}$ and $I_n = [T^*T]_\beta = A^tA$. Then $1 = \det(A^tA) = \det(A)^2$. Hence $\det(T) = \pm 1$.

Since T is orthogonal, $T(\sigma) = \{ T(e_1), T(e_2) \}$ is also an orthonormal basis for $\mathbf{R}^2$. Then $T(e_1)$ is a unit vector and there is θ such that $T(e_1) = (\cos(\theta), \sin(\theta))$. There are two possibilities for $T(e_2)$. Either $(-\sin(\theta), \cos(\theta))$ or $-(-\sin(\theta), \cos(\theta))$. Only the former gives $\det(A) = 1$ and then $A = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$ and T is rotation by θ. □

In Truth Is There No Beauty?

Example

A general quadratic equation in two variables is of the form

$$f(x,y) = ax^2 + 2bxy + cy^2 + dx + ey + f.$$

We are interested in the graph of $f(x,y) = 0$. If $b = 0$ and $ac \neq 0$, then we can "complete the square" twice so that

$$ax^2 + cy^2 + dx + ey + r = a(x - x_0)^2 + c(y - y_0)^2 + f'.$$

So we may as well concentrate on the equation

$$ax^2 + 2bxy + cy^2,$$

and try to eliminate the "xy"-term.

Getting Rid of xy

Using matrix multiplication, we can write

$$ax^2 + 2bxy + cy^2 = X^t AX = \langle AX\,,X\rangle,$$

where

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \quad \text{and} \quad X = \begin{pmatrix} x \\ y \end{pmatrix}:$$

Just check that $\left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} ax+by \\ bx+cy \end{smallmatrix}\right)$ and compute the (standard) inner product with $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$.

Since A is symmetric, it can be orthogonally diagonalized. Thus there is an orthogonal matrix Q such that $D = Q^t AQ$ is diagonal. This means that the columns of Q form an orthonormal basis $\beta = \{ v_1, v_2 \}$ of eigenvectors for $\mathbf{R}^2$. Let $D = \left(\begin{smallmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{smallmatrix}\right)$ where λ_k is the eigenvalue for v_k. If $\det(Q) = 1$, then Q is a rotation. If instead, $\det(Q) = -1$, then we can simply interchange v_1 and v_2 so that Q is indeed a rotation.

New Coordinates

Now we work in β coordinates. If $X = (x, y) \in \mathbf{R}^2$ we let $X' = (x', y') = [(x, y)]_\beta = [I]_\sigma^\beta X = Q^{-1}X = Q^t X$.

Since we made sure that Q was a rotation matrix, Q^t is also a rotation matrix and the β-coordinates (x', y') are obtained from (x, y) simply by rotation around the origin by a fixed angle θ.

Then $X = QX'$, and

$$\begin{aligned} X^t AX &= (QX')^t A(QX') = (X')^t Q^t AQX' = (X')^t DX' \\ &= \lambda_1 (x')^2 + \lambda_2 (y')^2. \end{aligned}$$

Classical Conics

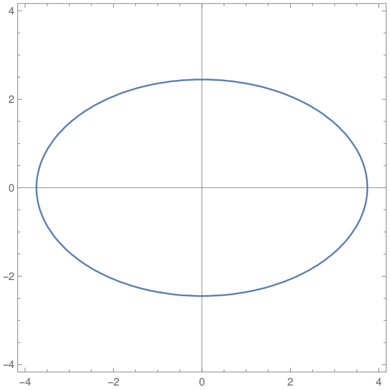

Figure: The ellipse $3x^2 + 7y^2 = 42$

Now we recall that equations of the form

$$\frac{x^2}{a_2} + \frac{y^2}{b^2} = 1 \quad a > 0 \text{ and } b > 0$$

give us an ellipse with axes $(-a, a)$ and $(-b, b)$. In particular, the equation $3x^2 + 7y^2 = 42$ is the ellipse drawn at left.

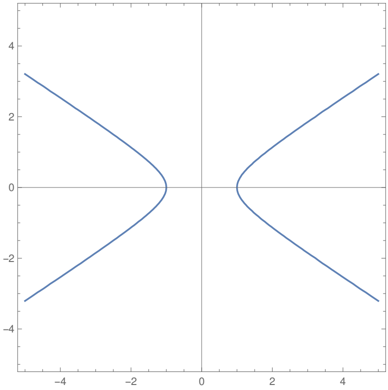

Figure: The Hyperbola
$3x^2 - 7y^2 = 3$

We also know that equations of the form

$$\frac{x^2}{a_2} - \frac{y^2}{b^2} = 1 \quad a > b > 0$$

give us a hyperbola with asymptotes $y = \pm\frac{b}{a}x$ In particular, the equation $3x^2 - 7y^2 = 3$ is the ellipse drawn at left.

Rotate

Example

Now consider the graph of $5x^2 - 4xy + 5y^2 = 42$. We need to eliminate the xy-term. Therefore we consider the symmetric matrix $A = \begin{pmatrix} 5 & -2 \\ -2 & 5 \end{pmatrix}$. The characteristic polynomial is $p(\lambda) = \lambda^2 - 10\lambda + 21$. Hence the eigenvalues are 3 and 7. You can check that $w_1 = (1, 1)$ and $w_2 = (-1, 1)$ are eigenvectors. Then $Q = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ satisfies $\det(Q) = 1$ and is rotation by $\frac{\pi}{4}$. Then

$$42 = 5x^2 - 4xy + 5y^2 = X^t AX = (X')^t DX' = 3(x')^2 + 7(y')^2$$

and we get our original ellipse rotated by $\frac{\pi}{4}$.

Picture

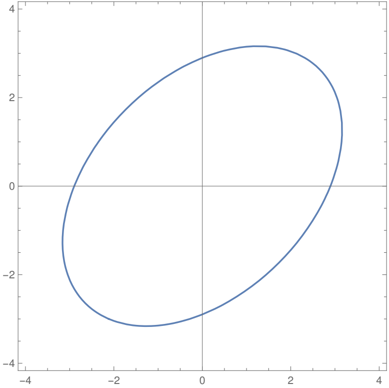

Figure: The Graph of $5x^2 - 4xy + 4y^2 = 42$

Example

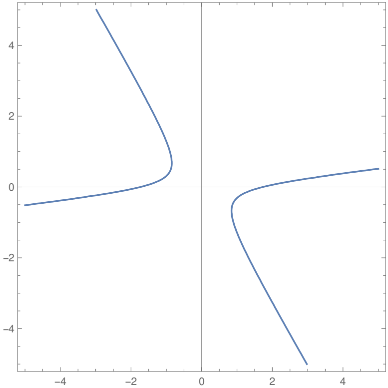

Figure: The Graph of $x^2 - 8xy - 5y^2 = 3$

Example

Now consider $x^2 - 8xy - 5y^2 = 3$. Here $A = \begin{pmatrix} 1 & -4 \\ -4 & -5 \end{pmatrix}$. We have $p(\lambda) = \lambda^2 + 4\lambda - 21$ with eigenvalues 3 and -7 and corresponding eigenvectors $w_1 = (2, -1)$ and $w_2 = (1, 2)$. Now $Q = \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}$ is rotation by $\arcsin(-\frac{1}{\sqrt{5}})$ (about -26 degress). And we get

$$\begin{aligned} 3 = x^2 - 8xy - 5y^2 &= X^t AX \\ &= (X')^t DX' = 3(x')^2 - 7(y')^2 \end{aligned}$$

Degenerate Example

Example

Now let's consider $x^2 + 4xy + 4y^2 + 2y = 4$. Again we start with $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$. The eigenvalues are 0 and 5 with eigenvectors $w_1 = (2, -1)$ and $w_2 = (1, 2)$. Then $Q = \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}$ is a rotation by $\arcsin(-\frac{1}{\sqrt{5}})$ (about -26 degress). Now in (x', y') coordinates, our equation is

$$5(y')^2 + 2(-\frac{1}{\sqrt{5}}x' + \frac{2}{\sqrt{5}}y') = 4.$$

Then after some algebra, this becomes

$$5(y' - \frac{2}{5\sqrt{5}})^2 = \frac{2}{\sqrt{5}}(x' + \frac{52}{5\sqrt{5}})$$

which is a parabola with vertex at $(-\frac{52}{5\sqrt{5}}, \frac{2}{5\sqrt{5}})$.

Picture

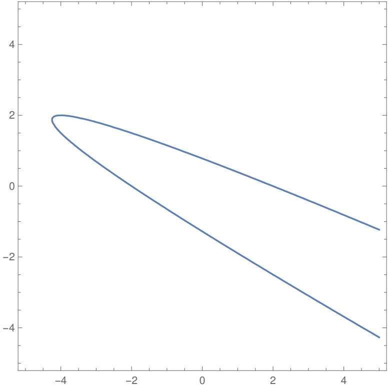

Figure: The Graph of $x^2 + 4xy + 4y^2 + 2y = 4$

Example

Of course we can do the similar things in higher dimensions. The quadratic equation in three variables

$$ax^2 + by^2 + cz^2 + 2dxy + 2eyz + 2fxz$$

can be written as X^tAX for $A = \begin{pmatrix} a & d & f \\ d & b & e \\ f & e & c \end{pmatrix}$. Since A is symmetric, we can find an orthonormal basis $\beta = \{ v_1, v_2, v_3 \}$ for $\mathbf{R}^3$ so that in β-coordinates we get

$$\lambda_1(x')^2 + \lambda_2(y')^2 + \lambda_3(z')^2.$$

But the details are best left for another day.

Enough

1. That is enough for Math 24.