WRITTEN HW #1, DUE OCT 3 2011

Remember to write clearly and to justify all your claims in your solutions.

(1) (10pts) Use induction to prove that

$$\sum_{k=1}^{n}(2k-1) = n^2.$$

(It is possible to prove the above formula without using induction, but for the purposes of this exercise use induction.)

(2) (10pts) The first two terms of the Fibonacci sequence are $F_1 = F_2 = 1$, and all succeeding terms are defined by the recurrence relation

$$F_{n+2} = F_{n+1} + F_n, n \geq 1.$$

A natural question to ask is whether there is an explicit formula for the general term of the Fibonacci sequence.

Consider the polynomial $x^2 - x - 1$. One easily checks that this has roots

$$\rho = \frac{1+\sqrt{5}}{2}, 1-\rho = \frac{1-\sqrt{5}}{2}.$$

Show that

$$F_n = \frac{\rho^n - (1-\rho)^n}{\sqrt{5}}.$$

(Incidentally, note that $\rho \approx 1.618$ is the golden ratio, a number which appears frequently in classical architechture.)

(3) (10pts) The first few rows of Pascal's triangle are

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 2 & & 1 & \\ 1 & & 3 & & 3 & & 1 \\ & & & \vdots & & & \end{array}$$

For reasons of notational convenience, we call the top row 'row 0', the next row 'row 1', etc. Each entry in a row is generated by summing the values of the two entries directly to the upper left and upper right of it. For example, the first 3 in the 3rd row is obtained by adding the two numbers immediately above it, which are 1 and 2. The first and last entries of each row are always 1.

Let n, k be integers satisfying $0 \leq k \leq n$. The binomial coefficient $\binom{n}{k}$, sometimes called n *choose* k, is defined by the equation

$$\binom{n}{k} = \frac{n!}{k!(n-k)!},$$

where $n! = n(n-1)(n-2)\dots(2)(1)$, called n *factorial*, is the product of the first n positive integers.

Show that the kth term of the nth row of Pascal's triangle is $\binom{n}{k}$. (Like the numbering of the rows themselves, we start the numbering of the elements in each row with 0, not 1.)

(4) (5pts) Use the Euclidean algorithm to compute $\gcd(a, b)$ for the following pairs a, b. Make sure to show each Euclidean division you perform.
  (a) $a = 255, b = 68$.
  (b) $a = 349, b = 17$.
  (c) $a = 196, b = 28$.

(5) (5pts) Use the Euclidean algorithm to find a pair of integer solutions (x, y) to

$$31x - 12y = 1.$$

(6) (10pts) Show that the product of any k consecutive positive integers is always divisible by $k!$.