Math 25

Homework 1

1. Use the Euclidean Algorithm to compute the gcd of 12345 and 67890.

2. Use your work above to find an integer solution to the equation $12345x + 67890y = 45$.

3. (a) Let $n \geq 2$ be an integer, and let $E_n = \{m \in \mathbb{Z} \mid \gcd(m, n) = 1\}$. Show that E_n is closed under multiplication, that is given $m, m' \in E_n$, show that $mm' \in E_n$.

   (b) Suppose that $a, b, c, d \in \mathbb{Z}$. Determine **necessary and sufficient conditions** on a, b, c, d so that the linear Diophantine equation $ax + by + cz = d$ is solvable for $x, y, z \in \mathbb{Z}$. *Hint:* There are a couple of cases to consider.

4. The following is an alternate (and more robust) definition for a greatest common divisor. Let $a, b \in \mathbb{Z}$, not both zero. We define a greatest common divisor of a, b (not necessarily unique) to be any integer d so that

$$\text{(i) } d \mid a,\ d \mid b, \text{ and (ii) if } c \mid a \text{ and } c \mid b, \text{ then } c \mid d.$$

   Notice that the difference between the two definitions is that we have replaced $\leq$ in the book's definition with $\mid$ (divides). Show that if d and d' are two greatest common divisors (as defined above), then $d' = \pm d$. As a consequence, there is a unique positive greatest common divisor which we call $\mathrm{GCD}(a, b)$.

5. Let $a, b \in \mathbb{Z}$, not both zero, and $\gcd(a, b)$ as defined in the book.

   (a) Show that $\gcd(a, b) = \mathrm{GCD}(a, b)$, so for elements in $\mathbb{Z}$, the two definitions coincide.

   (b) Show that if $c \in \mathbb{Z}$, then $c \mid a$ and $c \mid b$ if and only if $c \mid \gcd(a, b)$.

6. In calculus, you learn that the partial sums of the harmonic series $S_n := 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ grow without bound as $n \to \infty$. The goal of this exercise is to show that $S_n \in \mathbb{Z}$ if and only if $n = 1$.

   (a) As a lemma, show that if $a, b, c, d \in \mathbb{Z}$ with $\gcd(a, b) = \gcd(c, d) = 1$, then

$$\frac{a}{b} + \frac{c}{d} \in \mathbb{Z} \text{ implies that } b = \pm d.$$

   (b) Now show that $S_n \in \mathbb{Z}$ if and only if $n = 1$. *Hint:* Let t be the largest integer with $2^t \leq n$, and show that for $n \geq 2$, the product $2^{t-1} S_n$ can be written as $a/b + 1/2$ where $a, b \in \mathbb{Z}$ and b is odd. You probably want to look at some examples to see how this comes about, and what exactly you need to prove.