Your Name Goes Here

**Math 25**
Homework 2

1. Find all integer solutions to $11305x + 455y = 105$.

2. Find all integer solutions to $15x + 35y + 21z = 1$. Hint: Your answer should have two free parameters.

3. The Fibonacci numbers are a sequence of recursively defined integers $F_0, F_1, F_2, \ldots$ where $F_0 = 0$, $F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 0$. A few values of $F_n$:

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $F_n$ | 0 | 1 | 1 | 2 | 3 | 5 | 8 | 13 | 21 | 34 | 55 |

   Their occurrence in nature is uncanny, from models of uncontrolled population growth, to phyllotaxis, to number of seeds in spirals of sunflowers.

   (a) For $n \geq 0$, show that $\gcd(F_{n+1}, F_n) = 1$.

   (b) Show that for $n \geq 2$, the Euclidean algorithm takes exactly $n$ steps to compute $\gcd(F_{n+2}, F_{n+1})$

   (c) Consider a **generating function** for the Fibonacci numbers:

$$G(x) = \sum_{n=0}^{\infty} F_n x^n = x + x^2 + F_3 x^3 + F_4 x^4 + F_5 x^5 + \cdots .$$

   Show that

$$G(x) = \frac{x}{1 - x - x^2}.$$

   Hint: Compute $G(x) - xG(x) - x^2 G(x)$.

   (d) Exploration (**not required**). Show that for $n \geq 0$, $F_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n)$, where

$$\alpha = \frac{1+\sqrt{5}}{2}, \text{ and } \beta = \frac{1-\sqrt{5}}{2}.$$

   *Hint:* Start by using partial fractions to express

$$G(x) = \frac{x}{1 - x - x^2} = \frac{-x}{x^2 + x - 1} = \frac{A}{x - \alpha_0} + \frac{B}{x - \beta_0}.$$

4. Let $a, b$ be positive integers, and suppose that $\gcd(a, b) = 1$. We want to investigate **non-negative** integer solutions to $ax + by = c$, that is, solutions with both $x, y \geq 0$. Even though $\gcd(a, b) = 1$, there is no guarantee of any non-negative solutions, especially when $c$ is small relative to $a, b$.

(a) Show that for $c = ab$, the only non-negative solutions are $(0, a)$ and $(b, 0)$.

(b) Now let $c > ab$. Show that there is always a non-negative solution to $ax + by = c$.

5. Prove that there are infinitely many primes of the form $6m + 5$ for $m$ a non-negative integer.

6. Let $m, n$ be relatively prime positive integers.

   (a) Prove that if an integer $m > 1$ factors as $m = p_1^{e_1} \cdots p_r^{e_r}$ and $d \mid m$, then $d$ can be written as $d = p_1^{a_1} \cdots p_r^{a_r}$ where $0 \leq a_i \leq e_i$ for $1 \leq i \leq r$.

   (b) Show that every divisor $d \mid mn$ can be written uniquely as $d = d_m d_n$ where $d_m \mid m$ and $d_n \mid n$.

   (c) Use the above to show that if $n \geq 2$ is an integer with prime factorization $n = p_1^{e_1} \cdots p_r^{e_r}$, then every divisor $d \mid n$ can be written uniquely as $d = p_1^{a_1} \cdots p_r^{a_r}$ where $0 \leq a_i \leq e_i$ for $i = 1, \ldots, r$.