Homework 1

(Euclidean algorithm, Bezout's identity, gcd, lcm, linear Diophantine equations)

Due Wednesday, September 25 at 11:30am in class.

Note: Be sure to justify your answers. No credit will be given for answers without work/justification. In addition, all written homework assignments should be neat and well-organized; **Part A and Part B should be submitted separately**.

Part A:

(1) For these problems, you may use a calculator to do the arithmetic.

(a) (3 points) Use the Euclidean algorithm to find $\gcd(12345, 67890)$. Then, find $\text{lcm}(12345, 67890)$.

(b) (5 points) Find the general solution of the Diophantine equation $27x + 399y = -12$ using methods from this class.

(2) (2 points each) An example showing that a mathematical statement is false is called a *counterexample*. A counterexample must satisfy the *conditions* of the statement ("if ... "), but not satisfy the *conclusion* ("then ...").

Find counterexamples to each of the following statements and justify that the statement is false for your example.

(a) Let a, b, c, and d be integers. If a divides b, and c divides d, then $a+c$ divides $b+d$.

(b) If a and b are integers, then $\gcd(a, b) = \gcd(a + b, a - b)$.

(c) Let $a, b, c \in \mathbb{Z}$. If a divides c and b divides c, then ab divides c.

(d) If a and b are integers, then $\gcd(a, b) \cdot \text{lcm}(a, b) = ab$.

Part B: (3 points each) Prove each of the following statements.

(1) Let a be a positive integer. Then, a is divisible by 4 if and only if its last two digits are divisible by 4. (For example, 516 is divisible by 4 because 16 is divisible by 4.)

(2) For every integer k, the numbers $2k + 1$ and $9k + 4$ are relatively prime.

(3) Prove that if you carry out two steps in the Euclidean algorithm for $\gcd(a, b)$ with $a > b > 0$, then the remainder is less than $a/2$.

Fun problem (will not be graded): The Euclidean algorithm works so well that it is difficult to find pairs of numbers that make it take a long time. Find two numbers whose gcd is 1, for which the Euclidean algorithm takes 10 steps.