Homework 2

(Fundamental Theorem of Arithmetic, distribution of primes, primality testing, modular arithmetic)

Due Wednesday, October 2 at 11:30am in class.

Note: Be sure to justify your answers. No credit will be given for answers without work/justification. In addition, all written homework assignments should be neat and well-organized; **this assignment has only one part and can be submitted as a single packet**.

(1) Consider the set of even integers $\mathbb{E}$. Let $a, b \in \mathbb{E}$. We say that a $\mathbb{E}$–divides b if there exists $c \in \mathbb{E}$ such that $ac = b$. An element p of $\mathbb{E}$ is called $\mathbb{E}$–prime if there is no element in $\mathbb{E}$ which $\mathbb{E}$–divides p. The following is a list of some $\mathbb{E}$–primes:

$$2, 6, 10, 14, 18, \ldots$$

(a) Describe all $\mathbb{E}$–primes.

(b) Suppose that $a, b \in \mathbb{E}$ and p is an $\mathbb{E}$–prime. Show, via a counterexample, that the following statement does *not* hold:
"If p $\mathbb{E}$–divides ab, then p $\mathbb{E}$–divides a or p $\mathbb{E}$–divides b."

(c) Prove that every positive element in $\mathbb{E}$ has a prime factorization.

(d) Show, via a counterexample, that prime factorization in $\mathbb{E}$ is not unique.

(2) (a) For which primes p is $p^2 + 2$ also prime?

(b) Show that if $p > 1$ and p divides $(p-1)! + 1$, then p is prime.

(c) Suppose that a positive integer n has k 0s at the end of its decimal expansion. (For example, if $n = 3100$, then $k = 2$.) What can you say about e, where $2^e \mid\mid n$?

(3) (a) Without using a calculator, find:

(i) the least non-negative residue of $30 \times 27 \mod (13)$.

(ii) the least absolute residue of $82 \times 63 \mod (55)$.

(iii) the remainder when 10^9 is divided by 7.

(iv) the final decimal digit of $2! + 4! + 6! + \cdots 12!$.

(b) Find and prove a divisibility rule for 7.
(Hint: Consider the powers of 10 modulo 7. Can you find a pattern?)

Fun problem (will not be graded): The Fibonacci sequence is given by $F_n = F_{n-1} + F_{n-2}$ where $F_0 = F_1 = 1$. For example, the first several terms in the sequence are

$$1, 1, 2, 3, 5, 8, 13, \ldots.$$

Use strong induction to show that there is a closed formula for the nth Fibonacci number, namely

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$