Induction practice problems with solutions

(1) Show that $1^2 + 2^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$ for every positive integer n.

Proof. We proceed by induction on n.
Base case: If $n = 1$, then $1^2 = 1 = \dfrac{1(2)(3)}{6}$, as desired.
Induction hypothesis (IH): Fix $n \geq 1$ and assume that

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Induction step: We want to show that

$$\begin{aligned} 1^2 + 2^2 + \cdots + (n+1)^2 &= \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6}. \\ &= \frac{(n+1)(n+2)(2n+3)}{6}. \end{aligned}$$

Note that

$$\begin{aligned} 1^2 + 2^2 + \cdots + (n+1)^2 &= (1^2 + 2^2 + \cdots + n^2) + (n+1)^2 \\ &= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \text{ by our IH} \\ &= \frac{n(n+1)(2n+1) + 6(n+1)^2}{6} \\ &= \frac{(n+1)(n(2n+1) + 6(n+1))}{6} \\ &= \frac{(n+1)(2n^2 + 7n + 6)}{6} \\ &= \frac{(n+1)(n+2)(2n+3)}{6}, \end{aligned}$$

as desired. □

(2) Show that $n^3 - n$ is divisible by 6 for all $n \in \mathbb{N}$.

Proof. We proceed by induction on n.

Base case: If n=1, we have that $1^3 - 1 = 0$, which is divisible by 6 (since $0 \div 6 = 0$).

Induction hypothesis (IH): Fix $n \geq 1$ and assume that $n^3 - n$ is divisible by 6.

Induction step: We want to show that $(n+1)^3 - (n+1)$ is divisible by 6. Distributing and rearranging, we have that

$$\begin{aligned}(n+1)^3 - (n+1) &= n^3 + 3n^2 + 2n \\ &= (n^3 - n) + 3n^2 + 3n.\end{aligned}$$

By our induction hypothesis, $6 \mid (n^3 - n)$ and so it remains to show that $6 \mid (3n^2 + 3n)$. Clearly, $3n^2 + 3n = 3(n^2 + n)$ is divisible by 3. So, if $n^2 + n$ is even, then $3n^2 + 3n$ is divisible by both 2 and 3, and we are done. To show that $n^2 + n$ is even, we consider two cases: either n is even or n is odd.

If n is even, then $n = 2k$ for some integer k. Then $n^2 + n = 4k^2 + 2k = 2(2k^2 + k)$, which is even.

If n is odd, then $n = 2k + 1$ some integer k. Then $n^2 + n = (2k+1)^2 + 2k + 1 = 4k^2 + 6k + 2 = 2(2k^2 + 3k + 1)$, which is even.

Thus $6 \mid (3n^2 + 3n)$ and $6 \mid (n^3 - n)$ and so $6 \mid ((n^3 - n) + 3n^2 + 3n)$, as desired. $\square$

(3) Show that $2^{n+2} + 3^{2n+1}$ is divisible by 7 for all positive integers n.

For this example, I will follow the framework of the induction proofs above, but written in paragraph form and without the headings. Both are acceptable practices.

Proof. We proceed by induction on n. If $n = 1$, then we have $2^3 + 3^3 = 35$ which is divisible by 7. Fix $n \geq 1$ and assume that $2^{n+2} + 3^{2n+1}$ is divisible by 7; in other words, $2^{n+2} + 3^{2n+1} = 7m$ for some integer m. We want to show that $2^{(n+1)+2} + 3^{2(n+1)+1}$ is divisible by 7. We have that

$$\begin{aligned} 2^{(n+1)+2} + 3^{2(n+1)+1} &= 2^{n+3} + 3^{2n+3} \\ &= 2 \cdot 2^{n+2} + 3^2 \cdot 3^{2n+1} \\ &= 2 \cdot (7m - 3^{2n+1}) + 9 \cdot 3^{2n+1} \\ &= 14m + 3^{2n+1}(-2 + 9) \\ &= 7(2m + 3^{2n+1}), \end{aligned}$$

as desired. Therefore, by induction, $2^{n+2} + 3^{2n+1}$ is divisible by 7 for all positive integers n. □