Chapter 3

Distribution Problems

3.1 The idea of a distribution

Many of the problems we solved in Chapter 1 may be thought of as problems of distributing objects (such as pieces of fruit or ping-pong balls) to recipients (such as children). Some of the ways of viewing counting problems as distribution problems are somewhat indirect. For example, in Problem 31 you probably noticed that the number of ways to pass out k ping-pong balls to n children is the number of ways that we may choose a k-element subset of an n-element set. We think of the children as recipients and objects we are distributing as the identical ping-pong balls, distributed so that each recipient gets at most one ball. Those children who receive an object are in our set. It is helpful to have more than one way to think of solutions to problems. In the case of distribution problems, another popular model for distributions is to think of putting balls in boxes rather than distributing objects to recipients. Passing out identical objects is modeled by putting identical balls into boxes. Passing out distinct objects is modeled by putting distinct balls into boxes.

3.1.1 The twenty-fold way

When we are passing out objects to recipients, we may think of the objects as being either identical or distinct. We may also think of the recipients as being either identical (as in the case of putting fruit into plastic bags in the grocery store) or distinct (as in the case of passing fruit out to children). We may restrict the distributions to those that give at least one object to

Table 3.1: An incomplete table of the number of ways to distribute k objects to n recipients, with restrictions on how the objects are received

The Twentyfold Way: A Table of Distribution Problems		
k objects and conditions on how they are received	n recipients and mathematical model for distribution	
	Distinct	Identical
1. Distinct no conditions	n^k functions	? set partitions ($\leq n$ parts)
2. Distinct Each gets at most one	$n^{\underline{k}}$ k-element permutations	1 if $k \leq n$; 0 otherwise
3. Distinct Each gets at least one	? onto functions	? set partitions (n parts)
4. Distinct Each gets exactly one	$k! = n!$ bijections	1 if $k = n$; 0 otherwise
5. Distinct, order matters	? ?	? ?
6. Distinct, order matters Each gets at least one	? ?	? ?
7. Identical no conditions	? ?	? ?
8. Identical Each gets at most one	$\binom{n}{k}$ subsets	1 if $k \leq n$; 0 otherwise
9. Identical Each gets at least one	? ?	? ?
10. Identical Each gets exactly one	1 if $k = n$; 0 otherwise	1 if $k = n$; 0 otherwise

each recipient, or those that give exactly one object to each recipient, or those that give at most one object to each recipient, or we may have no such restrictions. If the objects are distinct, it may be that the order in which the objects are received is relevant (think about putting books onto the shelves in a bookcase) or that the order in which the objects are received is irrelevant (think about dropping a handful of candy into a child's trick or treat bag). If we ignore the possibility that the order in which objects are received matters, we have created $2 \cdot 2 \cdot 4 = 16$ distribution problems. In the cases where a recipient can receive more than one distinct object, we also have four more problems when the order objects are received matters. Thus we have 20 possible distribution problems.

We describe these problems in Table 3.1. Since there are twenty possible distribution problems, we call the table the "Twentyfold Way," adapting terminology suggested by Joel Spencer for a more restricted class of distribution problems. In the first column of the table we state whether the objects are distinct (like people) or identical (like ping-pong balls) and then give any conditions on how the objects may be received. The conditions we consider are whether each recipient gets at most one object, whether each recipient gets at least one object, whether each recipient gets exactly one object, and whether the order in which the objects are received matters. In the second column we give the solution to the problem and the name of the mathematical model for this kind of distribution problem when the recipients are distinct, and in the third column we give the same information when the recipients are identical. We use question marks as the answers to problems we have not yet solved and models we have not yet studied. We give explicit answers to problems we solved in Chapter 1 and problems whose answers are immediate. The goal of this chapter is to develop methods that will allow us to fill in the table with formulas or at least quantities we know how to compute, and we will give a completed table at the end of the chapter. We will now justify the answers that are not question marks.

If we pass out k distinct objects (say pieces of fruit) to n distinct recipients (say children), we are saying for each object which recipient it goes to. Thus we are defining a function from the set of objects to the recipients. We saw the following theorem in Problem 21c.

Theorem 3 *There are n^k functions from a k-element set to an n-element set.*

We proved it in Problem 21e. If we pass out k distinct objects (say pieces of fruit) to n indistinguishable recipients (say identical paper bags) then we are dividing the objects up into disjoint sets; that is we are forming a partition of the objects into some number, certainly no more than the number k of objects, of parts. Later in this chapter (and again in the next chapter) we shall discuss how to compute the number of partitions of a k-element set into n parts. This explains the entries in row one of our table.

If we pass out k distinct objects to n recipients so that each gets at most one, we still determine a function, but the function must be one-to-one. The number of one-to-one functions from a k-element set to an n element set is the same as the number of one-to-one functions from the set $[k] = \{1, 2, \ldots, k\}$ to an n-element set. In Problem 24 we proved the following theorem.

Theorem 4 *If $0 \le k \le n$, then the number of k-element permutations of an*

n-element set is

$$n^{\underline{k}} = n(n-1)\cdots(n-k+1) = n!/(n-k)!.$$

If $k > n$ there are no one-to-one functions from a k element set to an n element, so we define $n^{\underline{k}}$ to be zero in this case. Notice that this is what the indicated product in the middle term of our formula gives us. If we are supposed distribute k distinct objects to n identical recipients so that each gets at most one, we cannot do so if $k > n$, so there are 0 ways to do so. On the other hand, if $k \leq n$, then it doesn't matter which recipient gets which object, so there is only one way to do so. This explains the entries in row two of our table.

If we distribute k distinct objects to n distinct recipients so that each recipient gets at least one, then we are counting functions again, but this time functions from a k-element set *onto* an n-element set. At present we do not know how to compute the number of such functions, but we will discuss how to do so later in this chapter and in the next chapter. If we distribute k identical objects to n recipients, we are again simply partitioning the objects, but the condition that each recipient gets at least one means that we are partitioning the objects into exactly n blocks. Again, we will discuss how compute the number of ways of partitioning a set of k objects into n blocks later in this chapter. This explains the entries in row three of our table.

If we pass out k distinct objects to n recipients so that each gets exactly one, then $k = n$ and the function that our distribution gives us is a bijection. The number of bijections from an n-element set to an n-element set is $n!$ by Theorem 4. If we pass out k distinct objects of n identical recipients so that each gets exactly 1, then in this case it doesn't matter which recipient gets which object, so the number of ways to do so is 1 if $k = n$. If $k \neq n$, then the number of such distributions is zero. This explains the entries in row four of our table.

We now jump to row eight of our table. We saw in Problem 31 that the number of ways to pass out k identical ping-pong balls to n children is simply the number of k-element subsets of a k-element set. In Problem 33 we proved the following theorem.

Theorem 5 *If $0 \leq k \leq n$, the number of k-element subsets of an n-element* *set is given by*

$$\binom{n}{k} = \frac{n^{\underline{k}}}{k!} = \frac{n!}{k!(n-k)!}.$$

We define $\binom{n}{k}$ to be 0 if $k > n$, because then there are no k-element subsets of an n-element set. Notice that this is what the middle term of the formula in the theorem gives us. This explains the entries of row 8 of our table.

In row 10 of our table, if we are passing out k identical objects to n recipients so that each gets exactly one, it doesn't matter whether the recipients are identical or not; there is only one way to pass out the objects if $k = n$ and otherwise it is impossible to make the distribution, so there are no ways of distributing the objects. This explains the entries of row 10 of our table. Several other rows of our table can be computed using the methods of Chapter 1.

3.1.2 Ordered functions

100. Suppose we wish to place k distinct books onto the shelves of a bookcase with n shelves. For simplicity, assume for now that all of the books would fit on any of the shelves. Since the books are distinct, we can think of them as the first book, the second book and so on, perhaps in a stack. How many places are there where we can place the first book? When we place the second book, if we decide to place it on the shelf that already has a book, does it matter if we place it to the left of the book that is already there? How many places are there where we can place the second book? Once we have some number of books placed, if we want to place a new book on a shelf that already has some books, does it matter whether we place the new book to the immediate left of a book already on the shelf? In how many ways may we place the ith book into the bookcase? In how many ways may we place all the books?

101. Suppose we wish to place the books in Problem 100 so that each shelf gets at least one book. Now in how many ways may we place the books? (Hint: how can you make sure that each shelf gets at least one book as you start out putting books on the shelves?)

The assignment of which books go to which shelves of a bookcase is simply a function from the books to the shelves. But a function doesn't determine which book sits to the left of which others on the shelf, and this information is part of how the books are arranged on the shelves. In other words, the order in which the shelves receive their books matters. Our function must thus assign an ordered list of books to each shelf. We will call such a function an ordered function. More precisely, an **ordered function** from a set S to a set T is a function that assigns an (ordered) list of elements of S to some, but not necessarily all, elements of T in such a way that each element of S appears on one and only one of the lists.[1] (Notice that although it is not the usual definition of a function from S to T, a function can be described as an assignment of subsets of S to some, but not necessarily all, elements of T so that each element of S is in one and only one of these subsets.) Thus the number of ways to place the books into the bookcase is the entry in the middle column of row 5 of our table. If in addition we require each shelf to get at least one book, we are discussing the entry in the middle column of row 6 of our table. An *ordered onto function* is one which assigns a list to each element of T.

3.1.3 Broken permutations and Lah numbers

102. In how many ways may we stack k distinct books into n identical boxes so that there is a stack in every box? (Hint: Imagine taking a stack of k books, and breaking it up into stacks to put into the boxes in the same order they were originally stacked. If you are going to use n boxes, in how many places will you have to break the stack up into smaller stacks, and how many ways can you do this?) (Alternate hint: How many different bookcase arrangements correspond to the same way of stacking k books into n boxes so that each box has at least one book?). The hints may suggest that you can do this problem in more than one way!

We can think of stacking books into identical boxes as partitioning the books and then ordering the blocks of the partition. This turns out not to be a useful computational way of visualizing the problem because the number of ways to order the books in the various stacks depends on the sizes of the stacks and not just the number of stacks. However this way of thinking actually led to the first hint in Problem 102. Instead of dividing a set up

[1]The phrase ordered function is not a standard one, because there is as yet no standard name for the result of an ordered distribution problem.

into nonoverlapping parts, we may think of dividing a *permutation* (thought of as a list) of our k objects up into n ordered blocks. We will say that a set of ordered lists of elements of a set S is a **broken permutation** of S if each element of S is in one and only one of these lists.[2] The number of broken permutations of a k-element set with n blocks is denoted by $L(n, k)$. The number $L(n, k)$ is called a *Lah Number*.

The Lah numbers are the solution to the question "In how many ways may we distribute k distinct objects to n identical recipients if order matters and each recipient must get at least one?" Thus they give the entry in row 6 and column 6 of our table. The entry in row 5 and column 6 of our table will be the number of broken permutations with less than or equal to n parts. Thus it is a sum of Lah numbers.

We have seen that ordered functions and broken permutations explain the entries in rows 5 and 6 or our table.

3.1.4 Compositions of integers

103. In how many ways may we put k identical books onto n shelves if each shelf must get at least one book?

104. A **composition** of the integer k into n parts is a list of n positive integers that add to k. How many compositions are there of an integer k into n parts?

105. Your answer in Problem 104 can be expressed as a binomial coefficient. This means it should be possible to interpret a composition as a subset of some set. Find a bijection between compositions of k into n parts and certain subsets of some set. Explain explicitly how to get the composition from the subset and the subset from the composition.

106. Explain the connection between compositions of k into n parts and the problem of distributing k identical objects to n recipients so that each recipient gets at least one.

The sequence of problems you just completed should explain the entry in the middle column of row 9 of our table of distribution problems.

[2]The phrase broken permutation is not standard, because there is no standard name for the solution to this kind of distribution problem.

3.1.5 Multisets

In the middle column of row 7 of our table, we are asking for the number of ways to distribute k identical objects (say ping-pong balls) to n distinct recipients (say children).

107. In how many ways may we distribute k identical books on the shelves of a bookcase with n shelves, assuming that any shelf can hold all the books?

108. A multiset chosen from a set S may be thought of as a subset with repeated elements allowed. For example the multiset of letters of the word Mississippi is $\{i, i, i, i, m, p, p, s, s, s, s\}$. To determine a multiset we must say how many times (including, perhaps, zero) each member of S appears in the multiset. The size of a multiset chosen from S is the total number of times any member of S appears. For example, the size of the multiset of letters of Mississippi is 11. What is the number of multisets of size k that can be chosen from an n-element set?

109. Your answer in the previous problem should be expressible as a binomial coefficient. Since a binomial coefficient counts subsets, find a bijection between subsets of something and multisets chosen from a set S.

110. How many solutions are there in nonnegative integers to the equation $x_1 + x_2 + \cdots + x_m = r$, where m and r are constants?

The sequence of problems you just completed should explain the entry in the middle column of row 7 of our table of distribution problems. In the next two sections we will give ways of computing the remaining entries.

3.2 Partitions and Stirling Numbers

We have seen how the number of partitions of a set of k objects into n blocks corresponds to the distribution of k distinct objects to n identical recipients. While there is a formula that we shall eventually learn for this number, it requires more machinery than we now have available. However there is a good method for computing this number that is similar to Pascal's equation. Now that we have studied recurrences in one variable, we will point out that

Pascal's equation is in fact a *recurrence in two variables*; that is it lets us compute $\binom{n}{k}$ in terms of values of $\binom{m}{i}$ in which either $m < n$ or $i < k$ or both. It was the fact that we had such a recurrence and knew $\binom{n}{0}$ and $\binom{n}{n}$ that let us create Pascal's triangle.

3.2.1 Stirling Numbers of the second kind

We use the notation $S(k, n)$ to stand for the number of partitions of a k element set with n blocks. For historical reasons, $S(k, n)$ is called a *Stirling number of the second kind*.

111. In a partition of the set $[k]$, the number k is either in a block by itself, or it is not. How does the number of partitions of $[k]$ with n parts in which k is in a block with other elements of $[k]$ compare to the number of partitions of $[k-1]$ into n blocks? Find a two variable recurrence for $S(n, k)$, valid for n and k larger than one.

112. What is $S(k, 1)$? What is $S(k, k)$? Create a table of values of $S(k, n)$ for k between 1 and 5 and n between 1 and k. This table is sometimes called *Stirling's Triangle (of the second kind)* How would you define $S(k, n)$ for the nonnegative values of k and n that are not both positive? Now for what values of k and n is your two variable recurrence valid?

113. Extend Stirling's triangle enough to allow you to answer the following question and answer it. (Don't fill in the rows all the way; the work becomes quite tedious if you do. Only fill in what you need to answer this question.) A caterer is preparing three bag lunches for hikers. The caterer has nine different sandwiches. In how many ways can these nine sandwiches be distributed into three identical lunch bags so that each bag gets at least one?

114. The question in Problem 113 naturally suggests a more realistic question; in how many ways may the caterer distribute the nine sandwich's into three identical bags so that each bag gets exactly three? Answer this question. (Hint, what if the question asked about six sandwiches and two distinct bags? How does having identical bags change the answer?)

115. In how many ways can we partition k items into n blocks so that we have k_i blocks of size i for each i? (Notice that $\sum_{i=1}^{k} k_i = n$ and $\sum_{i=1}^{k} ik_i = k$.)

116. Describe how to compute $S(n,k)$ in terms of quantities given by the formula you found in Problem 115.

117. Find a recurrence similar to the one in Problem 111 for the Lah numbers $L(n,k)$.

118. The total number of partitions of a k-element set is denoted by $B(k)$ and is called the k-th *Bell number*. Thus $B(1) = 1$ and $B(2) = 2$.

    (a) Show, by explicitly exhibiting the partitions, that $B(3) = 5$.

    (b) Find a recurrence that expresses $B(k)$ in terms of $B(n)$ for $n \leq k$ and prove your formula correct in as many ways as you can.

    (c) Find $B(k)$ for $k = 4, 5, 6$.

3.2.2 Stirling Numbers and onto functions

119. Given a function f from a k-element set K to an n-element set, we can define a partition of K by putting x and y in the same block of the partition if and only if $f(x) = f(y)$. How many blocks does the partition have if f is onto? How is the number of functions from a k-element set onto an n-element set relate to a Stirling number? Be as precise in your answer as you can.

120. Each function from a k-element set K to an n-element set N is a function from K onto *some* subset of N. If J is a subset of N of size j, you know how to compute the number of functions that map onto J in terms of Stirling numbers. Suppose you add the number of functions mapping onto J over all possible subsets J of N. What simple value should this sum equal? Write the equation this gives you.

121. In how many ways can the sandwiches of Problem 113 be placed into three distinct bags so that each bag gets at least one?

122. In how many ways can the sandwiches of Problem 114 be placed into distinct bags so that each bag gets exactly three?

123. (a) How many functions are there from a set K with k elements to a set N with n elements so that for each i from 1 to n, k_i elements of N are each the images of i different elements of K. (Said differently, we have k_1 elements of N that are images of one element of K, we have k_2 elements of N that are images of two elements of K, and in general k_i elements of N that are images of i elements of K.) (We say y is the image of x if $y = f(x)$.)

(b) How many functions are there from a k-element set K to a set $N = \{y_1, y_2, \ldots y_n\}$ so that y_i is the image of j_i elements of K for each i from 1 to n. This number is called a *multinomial coefficient* and denoted by $\binom{k}{j_1, j_2, \ldots, j_n}$.

(c) Explain how to compute the number of functions from a k-element set K onto an n-element set N by using multinomial coefficients.

(d) What do multinomial coefficients have to do with expanding the kth power of a multinomial $x_1 + x_2 + \cdots + x_n$? This result is called the *multinomial theorem*

3.2.3 Stirling Numbers and bases for polynomials

124. Find a way to express n^k in terms of $k^{\underline{j}}$ for appropriate values j. Notice that $x^{\underline{j}}$ makes sense for a numerical variable x (that could range over the rational numbers, the real numbers, or even the complex numbers instead of only the nonnegative integers, as we are implicitly asuming n does), just as x^j does. Find a way to express the power x^k in terms of the polynomials $x^{\underline{j}}$ for appropriate values of j and explain why your formula is correct.

You showed in Problem 124 how to get each power of x in terms of the falling factorial powers $x^{\underline{j}}$. Therefore every polynomial in x is expressible in terms of a sum of numerical multiples of falling factorial powers. We say that the ordinary powers of x and the falling factorial powers of x are each bases for the "space" of polynomials, and that the numbers $S(k, n)$ are "change of basis coefficients."

125. Show that every power of $x + 1$ is expressible as a sum of powers of x. Now show that every power of x (and thus every polynomial in x) is a sum of numerical multiples (some of which could be negative) of powers

of $x+1$. This means that the powers of $x+1$ are a basis for the space of polynomials as well. Describe the change of basis coefficients that we use to express the binomial powers $(x+1)^n$ in terms of the ordinary x^j explicitly. Find the change of basis coefficients we use to express the ordinary powers x^n in terms of the binomial powers $(x+1)^k$.

126. By multiplication, we can see that every falling factorial polynomial can be expressed as a sum of numerical multiples of powers of x. In symbols, this means that there are numbers $s(k,n)$ (notice that this s is lower case, not upper case) such that we may write $x^{\underline{k}} = \sum_{n=0}^{k} s(k,n)x^n$. These numbers $s(k,n)$ are called Stirling Numbers of the first kind. By thinking algebraically about what the formula

$$x^{\underline{n}} = x^{\underline{n-1}}(x-n+1) \tag{3.1}$$

means, we can find a recurrence for Stirling numbers of the first kind that gives us another triangular array of numbers called Stirling's triangle of the first kind. Explain why Equation 3.1 is true and use it to derive a recurrence for $s(k,n)$ in terms of $s(k-1,n-1)$ and $s(k-1,n)$.

127. Write down the rows of Stirling's triangle of the first kind for $k=0$ to 6.

Notice that the Stirling numbers of the first kind are also change of basis coefficients. The Stirling numbers of the first and second kind are change of basis coefficients from the falling factorial powers of x to the ordinary factorial powers, and vice versa.

128. Explain why every rising factorial polynomial $x^{\overline{k}}$ can be expressed in terms of the falling factorial polynomials $x^{\underline{n}}$. Let $b(k,n)$ stand for the change of basis coefficients that allow us to express $x^{\overline{k}}$ in terms of the falling factorial polynomials $x^{\underline{n}}$; that is, define $b(k,n)$ by the equations

$$x^{\overline{k}} = \sum_{n=0}^{k} x^{\underline{n}}.$$

   (a) Find a recurrence for $b(k,n)$.

   (b) Find a formula for $b(k,n)$ and prove the correctness of what you say in as many ways as you can.

(c) Is $b(k, n)$ the same as any of the other families of numbers (binomial coefficients, Bell numbers, Stirling numbers, Lah numbers, etc.) we have studied?

(d) Say as much as you can (but say it precisely) about the change of basis coefficients for expressing $x^{\underline{k}}$ in terms of $x^{\overline{n}}$.

3.3 Partitions of Integers

We have now completed all our distribution problems except for those in which both the objects and the recipients are identical. For example, we might be putting identical apples into identical paper bags. In this case all that matters is how many bags get one apple (how many recipients get one object), how many get two, how many get three, and so on. Thus for each bag we have a number, and the multiset of numbers of apples in the various bags is what determines our distribution of apples into identical bags. A multiset of positive integers that add to n is called a **partition** of n. Thus the partitions of 3 are 1+1+1, 1+2 (which is the same as 2+1) and 3. The number of partitions of k is denoted by $P(k)$; in computing the partitions of 3 we showed that $P(3) = 3$.

129. Find all partitions of 4 and find all partitions of 5, thereby computing $P(4)$ and $P(5)$.

3.3.1 The number of partitions of k into n parts

130. A *partition of the integer k into n parts* is a multiset of n positive integers that add to k. We use $P(k, n)$ to denote the number of partitions of k into n parts. Thus $P(k, n)$ is the number of ways to distribute k identical objects to n identical recipients so that each gets at least one. Find $P(6, 3)$ by finding all partitions of 6 into 3 parts. What does this say about the number of ways to put six identical apples into three identical bags so that each bag has at least one apple?

131. With the binomial coefficients, with Stirling numbers of the second kind, and with the Lah numbers, we were able to find a recurrence by asking what happens to our subset, partition, or broken permutation of a set S of numbers if we remove the largest element of S. Thus it

is natural to look for a recurrence to count the number of partitions of k into n parts by doing something similar. However since we are counting distributions in which all the objects are identical, there is no way for us to identify a largest element. However we can ask what happens to a partition of an integer when we remove its largest part, or if we remove one from every part.

(a) How many parts does the remaining partition have when we remove the largest part from a partition of k into n parts? What can you say about the number of parts of the remaining partition if we remove one from each part.

(b) If the largest part of a partition is j and we remove it, what integer is being partitioned by the remaining parts of the partition? If we remove one from each part of a partition of k into n parts, what integer is being partitioned by the remaining parts?

(c) Use the answers to the last two questions to find and describe a bijection between partitions of k into n parts and partitions of smaller integers into appropriate numbers of parts.

(d) Find a recurrence (which need not have just two terms on the right hand side) that describes how to compute $P(k, n)$ in terms of the number of partitions of smaller integers into a smaller number of parts.

(e) What is $P(k, 1)$ for a positive integer k?

(f) What is $P(k, k)$ for a positive integer k?

(g) Use your recurrence to compute a table with the values of $P(k, n)$ for values of k between 1 and 7.

It is remarkable that there is no known formula for $P(k, n)$, nor is there one for $P(k)$. This section and some future parts of these notes are devoted to developing methods for computing values of $p(n, k)$ and finding properties of $P(n, k)$ that we can prove even without knowing a formula.

3.3.2 Representations of partitions

132. How many solutions are there in the positive integers to the equation $x_1 + x_2 + x_3 = 7$ with $x_1 \geq x_2 \geq x_3$?

133. Explain the relationship between partitions of k into n parts and lists $x_1, x_2, \ldots, x_n$ of positive integers with $x_1 \geq x_2 \geq \ldots \geq x_n$. Such a representation of a partition is called a *decreasing list* representation of the partition.

134. Describe the relationship between partitions of k and lists or vectors $(x_1, x_2, \ldots, x_n)$ such that $x_1 + 2x_2 + \ldots kx_k = k$. Such a representation of a partition is called a *type vector* representation of a partition, and it is typical to leave the trailing zeros out of such a representation; for example $(2, 1)$ stands for the same partition as $(2, 1, 0, 0)$. What is the decreasing list representation for this partition, and what number does it partition?

135. How does the number of partitions of k relate to the number of partitions of $k + 1$ whose smallest part is one?

3.3.3 Ferrers and Young Diagrams and the conjugate of a partition

The decreasing list representation of partitions leads us to a handy way to visualize partitions. Given a decreasing list $(k_1, k_2, \ldots k_n)$, we draw a figure made up of rows of dots that has k_1 equally spaced dots in the first row, k_2 equally spaced dots in the second row, starting out right below the beginning of the first row and so on. Equivalently, instead of dots, we may use identical squares, drawn so that a square touches each one to its immediate right or immediately below it along an edge. See Figure 3.1 for examples. The figure we draw with dots is called the Ferrers diagram of the partition; sometimes the figure with squares is also called a Ferrers diagram; sometimes it is called a Young diagram. At this stage it is irrelevant which name we choose and which kind of figure we draw; in more advanced work the boxes are handy because we can put things like numbers or variables into them. From now on we will use boxes and call the diagrams Young diagrams.

136. Draw the Young diagram of the partition (4,4,3,1,1). Describe the geometric relationship between the Young diagram of (5,3,3,2) and the Young diagram of (4,4,3,1,1).

137. The partition $(k_1, k_2, \ldots, k_n)$ is called the *conjugate* of the partition $(j_1, j_2, \ldots, j_m)$ if we obtain the Young diagram of one from the Young

Figure 3.1: The Ferrers and Young diagrams of the partition (5,3,3,2)

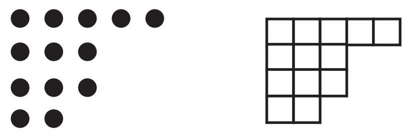

diagram of the other by flipping one around the line with slope -1 that extends the diagonal of the top left box. What is the conjugate of (4,4,3,1,1)? How is the largest part of a partition related to the number of parts of its conjugate? What does this tell you about the number of partitions of a positive integer k with largest part m?

138. A partition is called *self-conjugate* if it is equal to its conjugate. Find a relationship between the number of self-conjugate partitions of k and the number of partitions of k into distinct odd parts.

139. Explain the relationship between the number of partitions of k into even parts and the number of partitions of k into parts of even multiplicity, i.e. parts which are each used an even number of times as in (3,3,3,3,2,2,1,1).

140. Show that $P(k, n)$ is at least $\frac{1}{n!}\binom{k-1}{n-1}$.

We have seen that the number of partitions of k into n parts is equal to the number of ways to distribute k identical objects to n recipients so that each receives at least one. If we relax the condition that each recipient receives at least one, then we see that the number of distributions of k identical objects to n recipients is $\sum_{i=1}^{n} P(k, i)$ because if some recipients receive nothing, it does not matter which recipients these are. This completes rows 7 and 8 of our table of distribution problems. The completed table is shown in Figure 3.2. There are quite a few theorems that you have proved which are summarized by Table 3.2. It would be worthwhile to try to write them all down!

Table 3.2: The number of ways to distribute k objects to n recipients, with restrictions on how the objects are received

The Twentyfold Way: A Table of Distribution Problems		
k objects and conditions on how they are received	n recipients and mathematical model for distribution	
	Distinct	Identical
1. Distinct no conditions	n^k functions	$\sum_{i=1}^{k} S(n,i)$ set partitions ($\leq n$ parts)
2. Distinct Each gets at most one	$n^{\underline{k}}$ k-element permutations	1 if $k \leq n$; 0 otherwise
3. Distinct Each gets at least one	$S(k,n)n!$ onto functions	$S(k,n)$ set partitions (n parts)
4. Distinct Each gets exactly one	$k! = n!$ permutations	1 if $k = n$; 0 otherwise
5. Distinct, order matters	$(k+n-1)^{\underline{k}}$ ordered functions	$\sum_{i=1}^{n} L(k,i)$ broken permutations ($\leq n$ parts)
6. Distinct, order matters Each gets at least one	$(k)^{\underline{n}}(k-1)^{\underline{k-n}}$ ordered onto functions	$L(k,n) = \binom{k}{n}(k-1)^{\underline{k-n}}$ broken permutations (n parts)
7. Identical no conditions	$\binom{n+k-1}{k}$ multisets	$\sum_{i=1}^{n} P(k,i)$ number partitions ($\leq n$ parts)
8. Identical Each gets at most one	$\binom{n}{k}$ subsets	1 if $k \leq n$; 0 otherwise
9. Identical Each gets at least one	$\binom{k-1}{n-1}$ compositions (n parts)	$P(k,n)$ number partitions (n parts)
10. Identical Each gets exactly one	1 if $k = n$; 0 otherwise	1 if $k = n$; 0 otherwise