Math 29: Numbers, Functions, and Sets

March 30th, 2022

1 Numbers

We work with the natural numbers, denoted by both ω and $\mathbb{N}$.

$$\omega = \mathbb{N} = \{0, 1, 2, 3, 4, \dots\}$$

In other words, the non-negative whole numbers. Notice the natural numbers are built up from zero by adding one each time. This is a **recursive** definition, which you will learn more about on Thursday.

While we will think about various objects, each of them can be represented in some way using natural numbers. When we assign a natural number n to some other object, we call n a **code** for that object. Take, for example, the integers.

$$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$$

We can represent these using the natural numbers as follows: the natural number n is an integer, but we can't use it as its own code, otherwise we wouldn't have any numbers left to code the negative integers. Instead, we let $2n$ code n, and for $n > 0$, we let $2n - 1$ code $-n$. In this way, every integer is uniquely associated with a natural number, and vice versa.

As another example, let F be a finite set of natural numbers. The **canonical code for** F is the natural number n whose binary expansion (as opposed to decimal expansion) is 1 in the i-th bit (i.e., digit) if $i \in F$, and 0 if $i \notin F$. In other words,

$$n = \sum_{i \in F} 2^i$$

We say $F = D_n$ to represent that F is the finite set coded by n.

There are plenty of other objects we will think about as being coded by natural numbers, however to talk about them we first need to talk about functions.

2 Functions

As in the rest of mathematics, we use the notation $f : A \to B$ to mean that f is a function with domain A and codomain B. In other words, for every element a of A, there is a unique element b of B associated with a by f, and we write $f(a) = b$ and say b is the image of a. If the range of f, the set of elements of B which are the image of something in A, is all of B, then we say f is **surjective**, or **onto**. If, whenever $f(a) = f(a')$, $a = a'$, then we say that f is **injective**, or **one-to-one**. That is, each element of the range of f is associated with a *unique* element of the domain. If f is both surjective and injective, then it is called **bijective**.

Unlike most of math, however, we allow **partial** functions. A **partial** function $f : A \to B$ need not be defined on all of A. If f does not give a value on a, then we say that $f(a)$ diverges, denoted by $f(a)\uparrow$. If there is b such that $f(a) = b$, then we say $f(a)$ converges and write $f(a)\downarrow$. In other words, the domain of f is a subset of A, just like the range of f is a subset of B. If $dom(f) = A$, then we say that f is **total**. (That is, in the traditional notation, all functions are considered total.)

Returning to the idea of coding other objects as natural numbers, the primary concern is that no two objects have the same code, and every object has a code. Given some set of X whose elements we would like to assign codes to, we would like a function $c : X \to \mathbb{N}$ which is total and injective. Surjectivity can be helpful, but isn't always necessary.

Lemma 1. *Let X be the set of all finite subsets of ω. Then coding maps described above, $z : \mathbb{Z} \to \omega$ and $f : X \to \omega$, are both bijections.*

Proof. Homework 1: Question 1. □

Given a positive natural number k and a set X, X^k represents the collection of ordered k-tuples of elements of X. In particular, ω^k is the set of k-tuples of natural numbers.

Consider the function $p : \omega^2 \to \omega$ defined via

$$p(i, j) = \frac{1}{2}(i + j)(i + j + 1) + j$$

This is called the **Cantor pairing function**

Lemma 2. *p is surjective.*

Proof: Given n, we need to find natural numbers x and y such that $p(x,y) = n$. Let k be the largest natural number such that $\frac{1}{2}(k)(k+1) \leq n$. Let $j = n - \frac{1}{2}(k)(k+1)$. As we are subtracting, we need to be careful that j is still a natural number. Since $\frac{1}{2}(k)(k+1) \leq n$, this difference is at least 0, so j is a natural number.

Let $i = k - j$. Similarly, we need to ensure that $j \leq k$. However, if $j > k$, then that means that

$$n = \frac{1}{2}(k)(k+1) + j \geq \frac{1}{2}(k)(k+1) + k + 1$$

Now by the following algebra we see that

$$\frac{1}{2}(k)(k+1) + k + 1 = (k+1)(\frac{k}{2} + 1) = \frac{1}{2}(k+1)(k+2)$$

But this contradicts the choice of k being maximal such that $n \geq \frac{1}{2}k(k+1)$, so we must have $j \leq k$. Thus i is an integer, and

$$p(i,j) = \frac{1}{2}(i+j)(i+j+1) + j = \frac{1}{2}k(k+1) + (n - \frac{1}{2}(k)(k+1) = n$$

As n was arbitrary, p is surjective.

Lemma 3. *p is injective.*

Proof: Suppose that $p(i,j) = n = p(x,y)$. In other words,

$$\frac{1}{2}(i+j)(i+j+1)+j = \frac{1}{2}(x+y)(x+y+1)+y$$

Then to show p is injective, we need to prove that $i = x$ and $j = y$. First, suppose $\frac{1}{2}(i+j)(i+j+1) < \frac{1}{2}(x+y)(x+y+1)$. This simplifies to

$$(i+j)(i+j+1) < (x+y)(x+y+1)$$

It follows that $i+j < x+y$, or equivalently (since we are dealing with natural numbers) $i+j+1 \leq x+y$, because the function k^2+k is increasing in the natural numbers by the first derivative test. But this is a contradiction of the assumption that $p(i,j) = p(x,y) = n$, as

$$\frac{1}{2}(i+j)(i+j+1)+j < \frac{1}{2}(i+j)(i+j+1)+i+j+1 =$$

$$\frac{1}{2}(i+j+1)(i+j+2) \leq \frac{1}{2}(x+y)(x+y+1) \leq \frac{1}{2}(x+y)(x+y+1)+y$$

Therefore, because of this contradiction, it must be the case that

$$\frac{1}{2}(i+j)(i+j+1) \geq \frac{1}{2}(x+y)(x+y+1)$$

which by symmetry implies

$$\frac{1}{2}(i+j)(i+j+1) = \frac{1}{2}(x+y)(x+y+1)$$

and thus $i+j = x+y$. Therefore, we can cancel the first terms from the equation $p(i,j) = p(x,y) = n$ to obtain $j = y$. It follows then that $i = x$ as well, which is what we needed to show.

We often use $\langle x, y\rangle$ to denote $p(x, y)$. We can use the pairing function to provide coding functions for larger tuples such as triples, quadruples, quintuples, etc.

Exercise 4. *Write down a polynomial coding function which provides a bijection between ω^4 and ω.*

Proof. Homework 1: Question 2. □

Exercise 5. *Let $p^k : \omega^k \to \omega$ be coding functions which assign a code to each k-tuple for each k. Let X be the set of all tuples of natural numbers of size at least 2. Give a bijection $g : X \to \omega$ in terms of the p^k's.*

Proof. Homework 1: Question 3. □

3 Sets

As long as we are considering objects coded by natural numbers, then sets containing said objects correspond to sets of natural numbers.

Given a set $A \subseteq \omega$, then the **characteristic function** of A, $\chi_A : \omega \to \{0, 1\}$ is the function such that

$$\chi_A(n) = \begin{cases} 1 & n \in A \\ 0 & n \notin A \end{cases}$$

If the elements of A are, in increasing order, $a_0 < a_1 < a_2 < \dots$, then the **principal function** of A is the function p_A such that $p_A(n) = a_n$. In other words, it sends n to the n-th element of A. If A is finite with k elements, then p_A is partial and $p_A(n)\uparrow$ for $n \geq k$.

Later in the course we will describe what it means for functions to be computable and partial computable. Then sets will be computable (respectively partial computable) if their characteristic or principle functions are.