Math 29: The Turing Jump

May 16th, 2022

1 Turing Jump

Recall the halting problems: H asks the question "for which e and n does $\varphi_e(n)$ converge?" Similarly, K asks the question "for which e does $\varphi_e(e)$ converge?" As with other statements about computatin, we can relativize the halting problem. For brevity, we will stick to K, but we can relativize H in exactly the same fashion.

Given a set X, the halting problem relative to X is denoted by X' and represents the set

$$\{e : \Phi_e^X(e) \downarrow\}$$

This is called the **Turing jump** of X, often shortened to just the jump. $K = \emptyset'$, and we will often call it "the jump." Technically K may not exactly equal $\emptyset'$, or in fact C' for any computable set C since our coding of oracle mchines may produce different codes for machines than the one for regular machines, but as we will see below this does not matter.

Lemma 1. *If $A \leq_T B$, then $A' \leq_T B'$.*

Proof: Given e, we need to determine if $\Phi_e^A(e) \downarrow$. As $A \leq_T B$, there is k such that $\chi_A = \Phi_k^B$. Then by the relativized s-m-n theorem, there is a total computable function f such that, given e, $f(e)$ is the code for the oracle machine which replaces the `oracle` function with Φ_k^X.

Then $\Phi_{f(m)}^B(n) = \Phi_m^A(n)$ for all m and n. Then $e \in A'$ if and only if $\langle f(e), e\rangle$ is in H relativized to B, H^B. Thus this proves that $A \leq_1 H^B$. Then by the relativization of the fact that $H \leq_1 K$, H relative to B is 1-reducible to B', and thus by transitivity $A' \leq_1 B'$. (And therefore $\leq_T$ as well.)

In particular, if $A \equiv_T B$, then $A' \equiv_T B'$, so which set we take the jump of does not matter: we will end up in the same Turing degree regardless.

The halting problem was our prototypical example of a noncomputable set. We can relativize this proof to show that there is no highest Turing degree.

Lemma 2. *For all X, $X <_T X'$. That is $X \leq_T X'$ but $X' \not\leq_T X$.*

Proof: We first show that $X' \not\leq_T X$. Assume for the sake of contradiction that $X \leq_T X'$. Then define the X-computable function

$$f(e) = \begin{cases} 1 & \text{if } e \notin X' \\ \uparrow & \text{if } e \in X' \end{cases}$$

As X is X'-computable, f is X-computable, and therefore it has some X-index i. Now consider whether or not $i \in X'$. From this, we obtaain the same contradiction as we did in the proof that K was not computable, so X' cannot be X-computable.

We now show that $X \leq_T X'$. By the relativized s-m-n theorem, there is a total computable function f such that

$$\Phi^X_{f(e)}(n) = \begin{cases} 0 & e \in X \\ \uparrow & \text{otherwise} \end{cases}$$

Then $f(e) \in X'$ if and only if $e \in X$, so f 1-reduces X to X'.

In fact, the same proof we used to show that $W_e \leq_1 K$ for all e relativizes to prove that $W^X_e \leq_1 X'$ for all X and all e, where $W^X_e = dom(\Phi^X_e)$. That is, every set which is c.e. relative to X is 1-reducible to X'. In particular, this allows us to invert the above lemma: If $A' \leq_1 B'$, then $A \leq_T B$.

This implies that there is no highest Turing degree: the jump always gets us strictly higher than where we started, and it is complete for the X-c.e. sets.

Nothing stops us from iterating the jump operation. $X'' = (X')'$, $X''' = ((X')')'$, and so on and so forth. We can also use the following notation: $X^{(0)} = X$ and $X^{(n+1)} = (X^{(n)})'$. That is $X^{(n)}$ is the n-th jump of X. By the above, each of these is not computable from the previous, so this gives us an infinite, ascending chain above every single degree.

A natural question one might ask: is every set computable from X' c.e. in X? Well, we know that this isn't true for a silly reason, as the complement of any noncomputable c.e. set is computable from the jump but is not c.e. However, we can make this question by talking about degrees. A Turing degree is a **c.e. degree** if it contains a c.e. set. Is every degree below the degree of X' an X-c.e. degree? (The other direction we proved above.)

In fact, we don't even know this for $\emptyset'$: is there a degree below K which is not a c.e. degree? It turns out the answer is yes.

Theorem 3. *There is a set $X \leq_T K$ which is not of c.e. degree.*

Proof: Note that the c.e. sets are uniformly computable in K: $g(n,k) = \chi_H(\langle n,k\rangle)$ is computable in K because H is, and uniformly computes the c.e. sets. We will now build a set $B \leq_T K$ such that $B \not\equiv_T W_e$.

Let $\sigma_0 = \emptyset$, the empty binary string. Given σ_{2s} for $s = \langle n,k\rangle$ define σ_{2s+1} as follows: The set

$$\{\tau : \sigma_{2s} \preceq \tau \text{ and } \exists m < |\tau|\ \Phi^{\tau}_{n,|\tau|}(m)\downarrow \neq \chi_{W_k}(m)\}$$

is computable, and since K is equivalent to the index set of nonempty c.e. sets, K can determine if it is empty or not. In other words, is there an extension of σ_{2s} which witnesses $\Phi^B_n \neq W_k$? If so, let $\sigma_{s+1} = \tau$, where τ is the first to be found. Otherwise, set $\sigma_{s+1} = \sigma_s 1$.

For σ_{2s+1}, define σ_{2s+2} as follows. Define

$$\sigma_{2s+2} = \sigma_{2s+1}(1 - \chi_{W_s}(|\sigma_{2s+1}|))$$

That is, add one bit to the end of σ_{2s+1}, and make it the opposite of the bit of W_s in the same position to ensure that B will not be equal to W_s in the end, and therefore not computable.

Finally, let $B = \bigcup_{s\in\omega} \sigma_s$. Then B will not be computable, as σ_{2s+2} ensures $B \neq W_s$ for all s. Furthermore, we claim that any c.e. set which B computes is in fact computable.

If $\Phi^B_e = \chi_{W_k}$, then there must have been no τ and no m extending $\sigma_{2\langle e,k\rangle}$ such that $\Phi^{\tau}_{e,|\tau|}(m)\downarrow \neq \chi_{W_e}(m)$. Therefore, for every τ extending $\sigma_{2\langle e,k\rangle}$, if $\Phi^{\tau}_e(m)$ converges, it converges to $\chi_{W_k}(m)$. Because $\Phi^B_e = \chi_{W_e}$, we know that there is a τ which converges, so we can compute W_e as follows: look for the first τ extending $\sigma_{2\langle e,k\rangle}$ such that $\Phi^{\tau}_{e,|\tau|}(m)\downarrow$, and return its value. Then by the above, it must equal $\chi_{W_k}(m)$, so this process shows W_k is computable.

Thus B is not computable, but computes no non-computable c.e. set, and therefore is not of c.e. degree. But it is computable from K because K can uniformly determine if extensions exist at every stage.