Math 29: Midterm - Take-Home Portion

Due May 4th

For each of the following questions, provide a complete, clear solution. Remember to make it obvious which problem you are solving in each solution. Virtual submissions are due by midnight on the due date, either via Gradescope or email. Physical solutions are due in class on the due date. You are only permitted to use the lecture notes, the course textbook, and my help as a resource - no collaboration, internet, or other sources.

1. Two sets A and B are said to be **computably inseparable** if there is no computable set X such that $A \subseteq X$ and $B \cap X = \emptyset$. Give two c.e. sets which are computably inseparable. (Hint: There are some we have discussed before.)

2. Is K a simple set? Justify your answer.

3. Prove that no nontrivial index set is immune.

4. By the s-m-n theorem, there is a total computable function $c : \omega \to \omega$ such that:

   - If we see $\varphi_{e,s}(k) \downarrow$, then $\varphi_{c(e),s}(i) \downarrow = 0$ for all $i < k$ such that $\varphi_{e,s}(i) \uparrow$.
   - If $\varphi_{c(e),s}(k)$ has not been defined by use of the first case, and we see $\varphi_{e,s}(k) \downarrow$, we set $\varphi_{c(e),s}(k)$ equal to 0 if $\varphi_e(k) \downarrow = 0$, and 1 otherwise.

   Then this listing will contain only functions whose ranges are a subset of $\{0, 1\}$.

   (a) Prove that the characteristic function of every computable set is in this list.

   (b) Prove that, if the characteristic function of X is in this list, X is computable.

   (c) Why does this not contradict Homework 3 Question 3?

5. Let M be the set

$$\{x : \forall y < x \ \varphi_x \neq \varphi_y\}$$

That is, M is the set of minimal indices of computable functions: the smallest indices which define a given (partial) computable function.

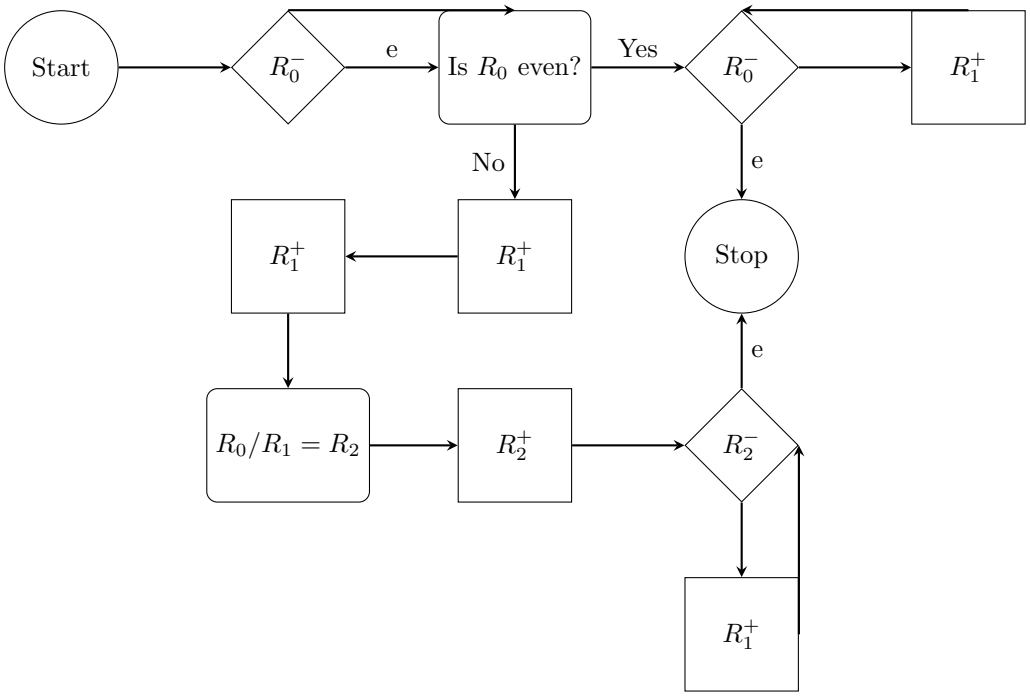
(a) Prove that M contains no infinite c.e. set.

(b) Is M productive? Creative?

(c) is M immune? Simple?

6. Consider the following register machine, which was programmed with the intention of computing

$$f(n) = \begin{cases} n-1 & \text{if n is odd} \\ \frac{n}{2} & \text{if n is even} \end{cases}$$



Assume every black box node answers accurately and fixes all registers.

(a) Explain why the register machine fails to compute f. (That is, why/when will the answer be wrong? Do not give a more efficient way to program the same machine.)

(b) Describe how to fix it, or provide an alternate register machine which computes f.