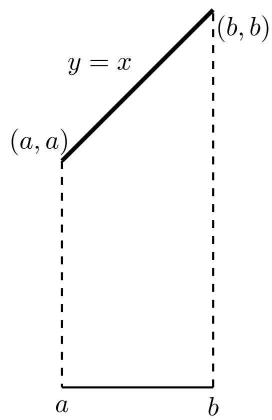
Math 2, Winter 2016

Daily Homework #12 — Solutions

4.2.27. *Prove that* $\displaystyle\int_a^b x\,dx = \frac{b^2 - a^2}{2}$.

Solution A. We can interpret the integral in terms of areas, since the region under the function is a trapezoid:



The right side's height is b, the left side's height is a, and the width of the bottom is $b - a$. Therefore, the area is $\displaystyle\int_a^b x\,dx = \frac{1}{2}(b + a)(b - a) = \frac{b^2 - a^2}{2}$.

Solution B. We use the definition of the integral as a limit of Riemann sums, with $f(x) = x$:

$$\begin{aligned}
\int_a^b x\,dx &= \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x^*)\,\Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(a + \frac{(b-a)i}{n}\right) \frac{b-a}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(a + \frac{(b-a)i}{n}\right) \frac{b-a}{n} \\
&= \lim_{n\to\infty} \left[\sum_{i=1}^{n} \left[a \cdot \frac{b-a}{n}\right] + \sum_{i=1}^{n} \left[\frac{(b-a)i}{n} \cdot \frac{b-a}{n}\right]\right] \\
&= \lim_{n\to\infty} \left[\frac{a(b-a)}{n} \left[\sum_{i=1}^{n} 1\right] + \frac{(b-a)^2}{n^2} \left[\sum_{i=1}^{n} i\right]\right] \\
&= \lim_{n\to\infty} \left[\frac{a(b-a)}{n} \cdot n + \frac{(b-a)^2}{n^2} \cdot \frac{n(n+1)}{2}\right] \\
&= \lim_{n\to\infty} \left[a(b-a) + \frac{(b-a)^2}{2} \cdot \frac{n(n+1)}{n^2}\right]
\end{aligned}$$

$$
\begin{aligned}
&= a(b-a) + \frac{(b-a)^2}{2} \\
&= ab - a^2 + \tfrac{1}{2}b^2 - ab + \tfrac{1}{2}a^2 \\
&= \tfrac{1}{2}b^2 - \tfrac{1}{2}a^2 \\
&= \frac{b^2 - a^2}{2}.
\end{aligned}
$$

Solution C. We use the fundamental theorem of calculus: since $\frac{1}{2}x^2$ is an antiderivative of x,

$$\int_a^b x\,dx = \tfrac{1}{2}x^2\Big]_{x=a}^{x=b} = \tfrac{1}{2}b^2 - \tfrac{1}{2}a^2 = \frac{b^2-a^2}{2}.$$

4.3.3. *Let $g(x) = \int_0^x f(t)\,dt$, where f is the function whose graph is shown.*

(a) *Evaluate $g(0)$, $g(1)$, $g(2)$, $g(3)$, and $g(6)$.*

Solution. Since g is the area function of f starting at $x = 0$, we can evaluate g by taking the integral, which we can interpret in terms of areas.

$$
\begin{aligned}
g(0) &= \int_0^0 f(t)\,dt = 0; \\
g(1) &= \int_0^1 f(t)\,dt = 2; \\
g(2) &= \int_0^2 f(t)\,dt = \int_0^1 f(t)\,dt + \int_1^2 f(t)\,dt = 2 + 3 = 5; \\
g(3) &= \int_0^3 f(t)\,dt = \int_0^2 f(t)\,dt + \int_2^3 f(t)\,dt = 5 + 2 = 7; \\
g(6) &= \int_0^6 f(t)\,dt = \int_0^3 f(t)\,dt + \int_3^6 f(t)\,dt = 7 - 4 = 3.
\end{aligned}
$$

(b) *On what interval is g increasing?*

Solution. By the fundamental theorem of calculus, f is the derivative of g. Therefore, g is increasing when f is positive, which is on the interval $(0, 3)$.

(c) *Where does g have a maximum value?*

Solution. Since g is continuous, and since it is increasing for $x < 3$ and decreasing for $x > 3$, it has an absolute maximum value at $x = 3$.

(d) *Sketch a rough graph of g.*

Solution. Again, f is the derivative of g; so g is concave up where f is increasing, and g is concave down where f is decreasing. This, along with the results of parts (a) through (c), helps us sketch the graph; see the back of the book for the picture.

4.3.8. *Use Part 1 of the Fundamental Theorem of Calculus to find the derivative of*

$$g(x) = \int_1^x \left(2 + t^4\right)^5 \, dt.$$

Solution. $\dfrac{dg}{dx} = \dfrac{d}{dx}\left[\displaystyle\int_1^x \left(2 + t^4\right)^5 dt\right] = \left(2 + x^4\right)^5.$

4.3.10. *Use Part 1 of the Fundamental Theorem of Calculus to find the derivative of*

$$g(r) = \int_0^r \sqrt{x^2 + 4} \, dx.$$

Solution. $\dfrac{dg}{dr} = \dfrac{d}{dr}\left[\displaystyle\int_0^r \sqrt{x^2 + 4} \, dx\right] = \sqrt{r^2 + 4}.$

4.3.22. *Evaluate* $\displaystyle\int_0^1 \left(1 + \tfrac{1}{2}u^4 - \tfrac{2}{5}u^9\right) du.$

Solution.

$$\begin{aligned}\int_0^1 \left(1 + \tfrac{1}{2}u^4 - \tfrac{2}{5}u^9\right) du &= \left(u + \tfrac{1}{10}u^5 - \tfrac{1}{25}u^{10}\right)\Big]_{u=0}^{u=1} \\ &= \left(1 + \tfrac{1}{10}1^5 - \tfrac{1}{25}1^{10}\right) - \left(0 + \tfrac{1}{10}0^5 - \tfrac{1}{25}0^{10}\right) = \frac{53}{50}.\end{aligned}$$

4.3.26. *Evaluate* $\displaystyle\int_{-5}^{5} \pi \, dx.$

Solution. Notice that π is a constant, so it has πx as an antiderivative. Thus

$$\int_{-5}^{5} \pi \, dx = \pi x\Big]_{x=-5}^{x=5} = 5\pi - (-5\pi) = 10\pi.$$

4.3.28. *Evaluate* $\displaystyle\int_0^4 (4 - t)\sqrt{t} \, dt.$

Solution.

$$\begin{aligned}\int_0^4 (4 - t)\sqrt{t} \, dt &= \int_0^4 (4 - t)t^{1/2} \, dt \\ &= \int_0^4 \left(4t^{1/2} - t^{3/2}\right) dt \\ &= \left(4 \cdot \tfrac{2}{3}t^{3/2} - \tfrac{2}{5}t^{5/2}\right)\Big]_{t=0}^{t=4} \\ &= \left(\tfrac{8}{3} - \tfrac{2}{5}t\right) t\sqrt{t}\,\Big]_{t=0}^{t=4} \\ &= \left(\tfrac{8}{3} - \tfrac{2}{5} \cdot 4\right) 4\sqrt{4} - \tfrac{8}{3} \cdot 0 = \frac{128}{15}.\end{aligned}$$