Math 2, Winter 2016

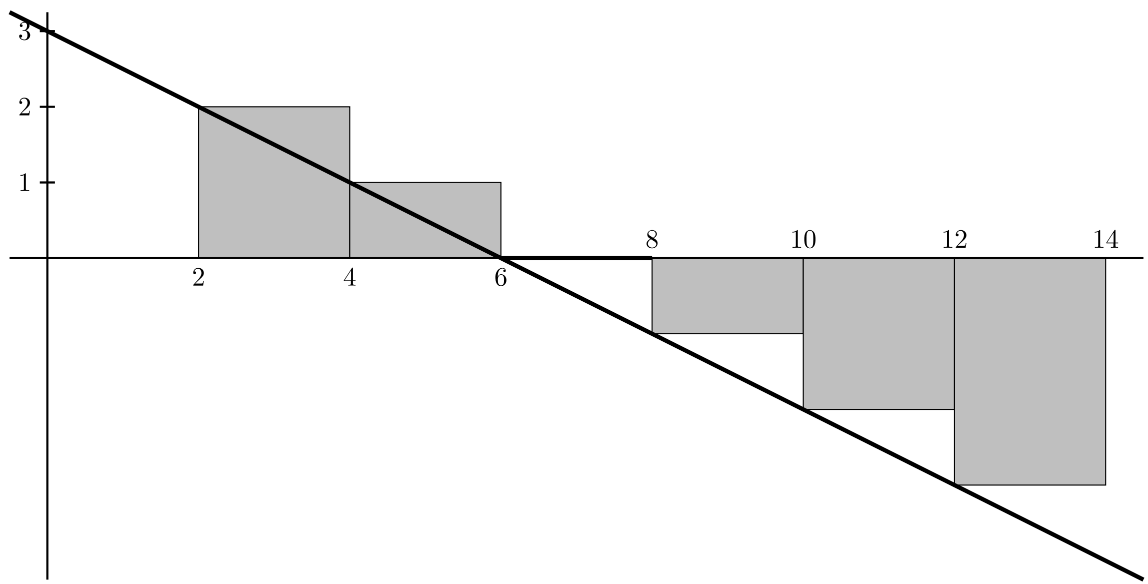
Daily Homework #13 — Solutions

4.2.1. *Evaluate the Riemann sum for* $f(x) = 3 - \frac{1}{2}x$, $x \le x \le 14$, *with six subintervals, taking the sample points to be left endpoints. Explain, with the aid of a diagram, what the Riemann sum represents.*

Solution. $\Delta x = \dfrac{b-a}{n} = \dfrac{14-2}{6} = 2$, so the Riemann sum is

$$\begin{aligned} f(2)\cdot 2 + f(4)\cdot 2 + f(6)\cdot 2 + f(8)\cdot 2 + f(10)\cdot 2 + f(12)\cdot 2 \\ = (2+1+0-1-2-3)\cdot 2 \\ = -6. \end{aligned}$$

So the Riemann sum equals -6. The Riemann sum represents the area under the line $f(x) = 3 - \frac{1}{2}x$; it is an approximation of $\int_2^{14} \left(3 - \frac{1}{2}x\right) dx$, as illustrated:



4.2.47. *Write as a single integral in the form* $\int_a^b f(x)\,dx$:

$$\int_{-2}^{2} f(x)\,dx + \int_{2}^{5} f(x)\,dx - \int_{-2}^{-1} f(x)\,dx$$

Solution. The first two pieces sum to $\int_{-2}^{5} f(x)\,dx$, and then taking away the third piece yields $\int_{-1}^{5} f(x)\,dx$.

4.2.55. *Use the properties of integrals to verify the inequality without evaluating the integral:*

$$\int_0^4 \left(x^2 - 4x + 4\right) dx \geq 0.$$

Solution. $x^2 - 4x + 4 = (x-4)^2$, which cannot be negative; so $x^2 - 4x + 4 \geq 0$. So,

$$\int_0^4 \left(x^2 - 4x + 4\right) dx \geq \int_0^4 0\, dx = 0.$$

4.3.31. *Evaluate* $\int_0^{\pi/4} \sec^2(t)\, dt$.

Solution. Recall that $\frac{d}{dt}\tan(t) = \sec^2(t)$, so $\tan(t)$ is an antiderivative of $\sec^2(t)$. So, using the Fundamental Theorem of Calculus,

$$\int_0^{\pi/4} \sec^2(t)\, dt = \tan(t)\Big]_{t=0}^{t=\pi/4} = \tan(\pi/4) - \tan(0) = 1 - 0 = 1.$$

4.3.34. *Evaluate* $\int_1^2 \frac{s^4+1}{s^2}\, ds$.

Solution.

$$\int_1^2 \frac{s^4+1}{s^2}\, ds = \int_1^2 \left(\frac{s^4}{s^2} + \frac{1}{s^2}\right) ds = \int_1^2 \left(s^2 + s^{-2}\right) ds$$
$$= \left(\tfrac{1}{3}s^3 - s^{-1}\right)\Big]_{s=1}^{s=2} = \left(\tfrac{1}{3}2^3 - \tfrac{1}{2}\right) - \left(\tfrac{1}{3}1^3 - \tfrac{1}{1}\right) = \frac{17}{6}.$$

4.3.38. *Evaluate* $\int_{-2}^2 f(x)\, dx$ *where* $f(x) = \begin{cases} 2 & \text{if } -2 \leq x \leq 0; \\ 4 - x^2 & \text{if } 0 < x \leq 2. \end{cases}$

Solution. We can break the integral into two pieces:

$$\int_{-2}^2 f(x)\, dx = \int_{-2}^0 2\, dx + \int_0^2 \left(4 - x^2\right) dx.$$

Now we use the Fundamental Theorem on each piece separately:

$$\int_{-2}^0 2\, dx = 2x\Big]_{x=-2}^{x=0} = 2(0) - 2(-2) = 4;$$

and

$$\int_0^2 \left(4 - x^2\right) dx = \left(4x - \tfrac{1}{3}x^3\right)\Big]_{x=0}^{x=2} = \left(4(2) - \tfrac{1}{3}(2^3)\right) - \left(4(0) - \tfrac{1}{3}(0^3)\right) = \frac{16}{3}.$$

So the total integral lis $4 + \frac{16}{3} = \frac{28}{3}$.

5.5.1. *Find the average value of* $f(x) = 4x - x^2$ *on* $[0, 4]$.

Solution.

$$\int_0^4 \left(4x - x^2\right) dx = \left(2x^2 - \tfrac{1}{3}x^3\right)\Big]_{x=0}^{x=4} = \left(2(4^2) - \tfrac{1}{3}(4^3)\right) - \left(2(0^2) - \tfrac{1}{3}(0^3)\right) = \frac{32}{3},$$

so the average value is $\dfrac{1}{4} \cdot \dfrac{32}{3} = \dfrac{8}{3}$.

5.5.2. *Find the average value of* $f(x) = \sin(4x)$ *on* $[-\pi, \pi]$.

Solution.

$$\int_{-\pi}^{\pi} \sin(4x)\, dx = -\tfrac{1}{4}\cos(4x)\Big]_{x=-\pi}^{x=\pi} = -\tfrac{1}{4}\cos(4\pi) - \left(-\tfrac{1}{4}\cos(-4\pi)\right) = -\tfrac{1}{4} + \tfrac{1}{4} = 0,$$

so the average value is $\dfrac{1}{2\pi} \cdot 0 = 0$.

5.5.15. *Find the average value of* f *(shown in a diagram) on* $[0, 8]$.

Solution. We can find $\int_0^8 f(x)\, dx$ by interpreting it as the combined areas of a bunch of triangles and rectangles in the usual way. Doing this results in $\int_0^8 f(x)\, dx = 9$, so the average value of f is $\dfrac{9}{8}$.