Math 2, Winter 2016

# Daily Homework #18 — Solutions

**9.2.3–6.** *Match the differential equation with its direction field. Give reasons for your answer.*

***3.*** $y' = 2 - y$; ***4.*** $y' = x(2-y)$; ***5.*** $y' = x + y - 1$; ***6.*** $y' = \sin(x)\sin(y)$.

**Solution.** For $y' = 2 - y$, the slope is positive for $y < 2$, zero for $y = 2$, and negative for $y > 2$. Also, the slope does not depend on $x$. Slope field III has these features.

For $y' = x(2-y)$, the slope is zero when $y = 2$ or when $x = 0$. Furthermore, when $x > 0$ and $y > 2$, the slope is negative. The only slope field that has these features is I.

For $y' = x + y - 1$, we see that the slope is negative below the line $y = -x + 1$, zero on this line, and positive above this line. This happens in slope field IV.

For $y' = \sin(x)\sin(y)$, we know it's II because that's the only option left. We also know it because the slope in 2 is zero when $x = 0$ or when $y = 0$, and $\sin(x)\sin(y)$ is zero at those points too (and other points, of course).

**9.3.1.** *Solve the differential equation* $\dfrac{dy}{dx} = xy^2$.

**Solution.**

$$\begin{aligned}
\frac{dy}{dx} &= xy^2;\\
\frac{dy}{y^2} &= x\,dx;\\
\int y^{-2}\,dy &= \int x\,dx;\\
-y^{-1} &= \tfrac{1}{2}x^2 + C;\\
y &= -\frac{1}{\frac{1}{2}x^2 + C}.
\end{aligned}$$

**9.3.2.** *Solve the differential equation* $\dfrac{dy}{dx} = xe^{-y}$.

**Solution.**

$$\begin{aligned}
\frac{dy}{dx} &= xe^{-y};\\
e^y\,dy &= x\,dx;\\
\int e^y\,dy &= \int x\,dx;\\
e^y &= \tfrac{1}{2}x^2 + C;\\
y &= \ln\left(\tfrac{1}{2}x^2 + C\right).
\end{aligned}$$

**9.3.11.** *Find the solution of* $\frac{dy}{dx} = \frac{x}{y}$ *that satisfies* $y(0) = -3$.

**Solution.**

$$\begin{aligned} \frac{dy}{dx} &= \frac{x}{y}; \\ y\,dy &= x\,dx; \\ \int y\,dy &= \int x\,dx; \\ \tfrac{1}{2}y^2 &= \tfrac{1}{2}x^2 + C; \\ y^2 &= x^2 + D \end{aligned}$$

(where we set $D = 2C$). We can find $D$ using the initial condition of $y(0) = -3$:

$$(-3)^2 = 0^2 + D \quad \Rightarrow \quad D = 9.$$

Now let's substitute this value of $D$, and solve for $y$:

$$\begin{aligned} y^2 &= x^2 + 9; \\ y &= \pm\sqrt{x^2 + 9}. \end{aligned}$$

Because $y(0) = -3$, this tells us to take the *negative* square root, and our solution to the initial value problem is

$$y = -\sqrt{x^2 + 9}.$$

**9.3.12.** *Find the solution of* $\frac{dy}{dx} = \frac{\ln(x)}{xy}$ *that satisfies* $y(1) = 2$.

**Solution.**

$$\begin{aligned} \frac{dy}{dx} &= \frac{\ln(x)}{xy}; \\ y\,dy &= \frac{\ln(x)}{x}\,dx; \\ \int y\,dy &= \int \frac{\ln(x)}{x}\,dx; \end{aligned}$$

The integral of $y$ is $\frac{1}{2}y^2$, but what is the integral of $\frac{\ln(x)}{x}$? We need to do a $u$ substitution: $u = \ln(x)$, and $du = \frac{1}{x}dx$. Then the integral becomes

$$\int \ln(x)\,\frac{1}{x}\,dx = \int u\,du = \tfrac{1}{2}u^2 + C = \tfrac{1}{2}\ln(x)^2 + C.$$

Then the integration by parts gives us

$$\tfrac{1}{2}y^2 = \tfrac{1}{2}\ln(x)^2 + C.$$

Now would be a good time to plug in the initial condition and solve for $C$. We are given $y(1) = 2$, so we get:

$$\begin{aligned} \tfrac{1}{2} \cdot 2^2 &= \tfrac{1}{2} \cdot \ln(1)^2 + C; \\ 2 &= 0 + C; \\ C &= 2. \end{aligned}$$

Now let's substitute this value of $C$, and solve for $y$:

$$\begin{aligned} \tfrac{1}{2} y^2 &= \tfrac{1}{2} \ln(x)^2 + 2; \\ y^2 &= \ln(x)^2 + 4; \\ y &= \pm\sqrt{\ln(x)^2 + 4}. \end{aligned}$$

Because $y(1) = 2$, this tells us to take the *positive* square root, and our solution to the initial value problem is

$$y = \sqrt{\ln(x)^2 + 3}.$$

**9.3.19.** *Find an equation of the curve that passes through the point* $(0, 1)$ *and whose slope at* $(x, y)$ *is* $xy$.

**Solution.** This means that $\dfrac{dy}{dx} = xy$ and $y(0) = 1$. Then

$$\begin{aligned} \frac{1}{y}\, dy &= x\, dx; \\ \int \frac{1}{y}\, dy &= \int x\, dx; \\ \ln|y| &= \tfrac{1}{2} x^2 + C. \end{aligned}$$

At this point we substitute the initial condition to find $C$:

$$\ln|1| = \tfrac{1}{2} 0^2 + C \quad \Rightarrow \quad 0 = 0 + C \quad \Rightarrow \quad C = 0.$$

Now we substitute this value of $C$ and solve for $y$:

$$\begin{aligned} \ln|y| &= \tfrac{1}{2} x^2; \\ |y| &= e^{\frac{1}{2}x^2}; \\ y &= \pm e^{\frac{1}{2}x^2}. \end{aligned}$$

Because $y(0) = 1$, this tells us to take the positive solution: $y = e^{\frac{1}{2}x^2}$.