

DAILY HOMEWORK #7 — SOLUTIONS

3.7.7. Find the dimensions of a rectangle with perimeter 100 m whose area is as large as possible.

Solution. Let x be the width (units m), y the height (units m), and A the area (units m^2). We want to maximize A . We know $A = xy$, and the perimeter is $100 = 2x + 2y$. Solving the perimeter equation for y in terms of x , we get $y = 50 - x$. We plug this into the area equation: $A = x(50 - x)$, and now A is a function of one variable.

What is the domain of A as a function of x ? We must have $x \geq 0$, and we also must have $y \geq 0$. If $y \geq 0$, then $50 - x \geq 0$, and solving for x yields $x \leq 50$. So the domain is $[0, 50]$.

We first simplify the area equation: $A = 50x - x^2$. Then we find the critical numbers: $\frac{dA}{dx} = 50 - 2x$, and this is never undefined, so we just set it to zero and solve for x :

$$50 - 2x = 0 \quad \Rightarrow \quad x = 25;$$

so 25 is the critical number. This number is in the domain, so to determine the maximum we test the original function on the critical number and on the endpoints:

$$A(0) = 0; \quad A(25) = 625; \quad A(50) = 0.$$

The highest of these is 625, so the dimensions are $x = 25$ and $y = 50 - 25 = 25$; that is, the rectangle is 25×25 . Notice that this is a square!

Alternatively, if you don't want to plug in the numbers at the end, you can tell that the critical number has the absolute maximum due to the fact that the graph of A is a parabola that opens downward (since A is a quadratic equation and the x^2 term is negative).

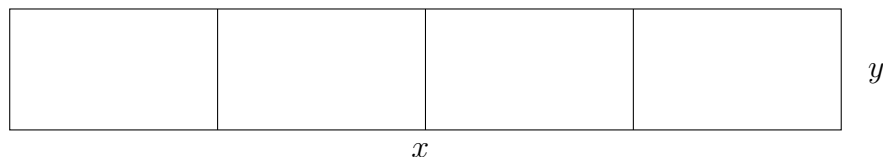
3.7.11. Consider the following problem: A farmer with 750 ft of fencing wants to enclose a rectangular area and then divide it into four pens with fencing parallel to one side of the rectangle. What is the largest possible total area of the four pens?

(a) Draw several diagrams illustrating the situation, some with shallow, wide pens and some with deep, narrow pens. Find the total areas of these configurations. Does it appear that there is a maximum area? If so, estimate it.

Solution. Illustrations of various arrangements of pens can be found on the last page of this solution set. My impression from drawing various arrangements is that the area is largest when each pen is close to being a square, but not quite a square. I would estimate that the width is 200 and the height is 70 (so each of the four pens is 50×70), which gives an area of $200 \cdot 70 = 14\,000$.

(b) Draw a diagram illustrating the general situation. Introduce notation and label the diagram with your symbols.

Solution. Call the width of the rectangle x (in feet), and call the height of the rectangle y (in feet). (You don't have to use these letters.) This is the diagram for the general situation, labeled with x and y :



(c) Write an expression for the total area.

Solution. $A = xy$.

(d) Use the given information to write an equation that relates the variables.

Solution. The total length of fence is 750 ft. There are two horizontal fences, each with length x ; and there are five vertical fences, each with length y ; so the equation we get is $750 = 2x + 5y$.

(e) Use part (d) to write the total area as a function of one variable.

Solution. We can solve for y in the equation in part (d): $y = 150 - \frac{2}{5}x$. Then plug this into the equation from (c): $A = x(150 - 0.4x)$.

(f) Finish solving the problem and compare the answer with your estimate in part (a).

Solution. What is the domain of A as a function of x ? We must have $x \geq 0$, and we also must have $y \geq 0$. If $y \geq 0$, then $150 - 0.4x \geq 0$, and solving for x yields $x \leq 375$. So the domain is $[0, 375]$.

We first simplify the equation from (e): $A = 150x - 0.4x^2$. Then we find the critical numbers: $\frac{dA}{dx} = 150 - 0.8x$, and this is never undefined, so we just set it to zero and solve for x :

$$150 - 0.8x = 0 \quad \Rightarrow \quad x = 187.5;$$

so 187.5 is the critical number. This number is in the domain, so to determine the maximum we test the original function on the critical number and on the endpoints:

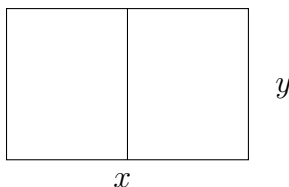
$$A(0) = 0; \quad A(187.5) = 14\,062.5; \quad A(375) = 0.$$

The highest of these is 14 062.5, so that is the largest possible area (in ft^2). Pretty close to my guess of 14 000.

Alternatively, if you don't want to plug in the numbers at the end, you can tell that the critical number has the absolute maximum due to the fact that the graph of A is a parabola that opens downward (since A is a quadratic equation and the x^2 term is negative).

3.7.13. A farmer wants to fence an area of 1.5 million square feet in a rectangular field and then divide it in half with a fence parallel to one of the sides of the rectangle. How can she do this so as to minimize the cost of the fence?

Solution. Let x be the width (units ft) and y the height (units ft). Here is a diagram of the situation:



Let L be the total length of fence (units ft). We want to minimize L . There are two horizontal fences, each with length x ; and there are three vertical fences, each with length y ; so the equation we get is $L = 2x + 3y$.

The area is $1\,500\,000 = xy$; solving this for y in terms of x , we get $y = \frac{1\,500\,000}{x}$. We plug this into the fence-length equation: $L = 2x + \frac{4\,500\,000}{x}$, and now L is a function of one variable.

What is the domain of L as a function of x ? We must have $x > 0$ and $y > 0$ (or else the area is zero instead of 1.5 million). Restricting $y > 0$ does not give any additional restriction on x . So the domain is $(0, \infty)$.

Now we take the derivative of L to find the critical numbers:

$$\frac{dL}{dx} = 2 - \frac{4\,500\,000}{x^2}.$$

This is undefined for $x = 0$, but this point is not in the domain, so we ignore it. Now we just set the derivative to zero and solve for x :

$$2 - \frac{4\,500\,000}{x^2} = 0 \quad \Rightarrow \quad 2x^2 - 4\,500\,000 = 0 \quad \Rightarrow \quad x^2 = 2\,250\,000 \quad \Rightarrow \quad x = \pm 1500.$$

Since -1500 is outside the domain, the only critical number is 1500 . Unfortunately the domain is not a closed interval, so we can't just test the endpoints in the original function.

However, we can see that $\frac{dL}{dx}$ is negative for all x in $(0, 1500)$, and positive for all x in $(1500, \infty)$; so in fact L has an absolute minimum at $x = 1500$ (using what the textbook calls the "first derivative test for absolute extreme values").

Then $y = \frac{1\,500\,000}{1500} = 1000$. So total fence length is minimized by making width 1500 ft and height 1000 ft.

3.7.19. Find the point on the line $y = 2x + 3$ that is closest to the origin.

Solution. By the distance formula, the distance from (x, y) to $(0, 0)$ is

$$D = \sqrt{x^2 + y^2}.$$

We want to minimize D for points (x, y) on the line $y = 2x + 3$. The equation of the line tells us y in terms of x , so we just plug in $y = 2x + 3$ in the equation for D :

$$D = \sqrt{x^2 + (2x + 3)^2}.$$

The line covers all x coordinates, so the domain is $(-\infty, \infty)$, *i.e.* all real numbers.

The minimum of D occurs at the same point as the minimum of D^2 , but D^2 is easier to work with; so we set

$$S = D^2 = x^2 + (2x + 3)^2,$$

and we set out to find the absolute minimum of S . Using the chain rule yields

$$\frac{dS}{dx} = 2x + 2(2x + 3) \cdot 2 = 10x + 12.$$

This is defined for all x , so we find the critical numbers by setting the derivative to zero: $10x + 12 = 0$, and then $x = -\frac{6}{5}$ is our critical number.

Unfortunately the domain is not a closed interval, so we can't just test the endpoints in the original function. However, we can see that $\frac{dS}{dx}$ is negative *for all* $x < -\frac{6}{5}$, and positive *for all* $x > -\frac{6}{5}$; so in fact S has an absolute minimum at $x = -\frac{6}{5}$ (using what the textbook calls the "first derivative test for absolute extreme values"). Alternatively, because S is a parabola that opens upward, the critical number has an absolute minimum.

Then $y = 2\left(-\frac{6}{5}\right) + 3 = \frac{3}{5}$. Therefore, the point on the line that lies closest to the origin is $\left(-\frac{6}{5}, \frac{3}{5}\right)$.