Math 2, Winter 2016

Daily Homework #9 — Solutions

3.3.13. (b) *Find the local maximum and minimum values of $f(x) = \sin(x) + \cos(x)$ on the interval $[0, 2\pi]$.*

Solution. We first find the critical numbers: $f'(x) = \cos(x) - \sin(x)$, and since this is never undefined we just set it to zero:

$$\cos(x) - \sin(x) = 0 \quad \Rightarrow \quad \sin(x) = \cos(x) \quad \Rightarrow \quad \frac{\sin(x)}{\cos(x)} = 1 \quad \Rightarrow \quad \tan(x) = 1;$$

and on the domain $[0, 2\pi]$ this is satisfied when $x = \frac{\pi}{4}$ or $x = \frac{5\pi}{4}$. Therefore, the critical numbers are $\frac{\pi}{4}$ and $\frac{5\pi}{4}$.

We can use the second-derivative test to classify each critical number as a local minimum or a local maximum (unless the test is inconclusive). We have $f''(x) = -\sin(x) - \cos(x)$, so

$$f''\left(\tfrac{\pi}{4}\right) = -\sin\left(\tfrac{\pi}{4}\right) - \cos\left(\tfrac{\pi}{4}\right) = -\tfrac{\sqrt{2}}{2} - \tfrac{\sqrt{2}}{2} = -\sqrt{2} < 0,$$

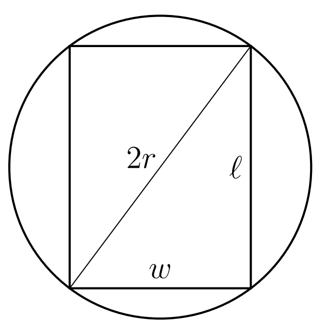
and

$$f''\left(\tfrac{5\pi}{4}\right) = -\sin\left(\tfrac{5\pi}{4}\right) - \cos\left(\tfrac{5\pi}{4}\right) = \tfrac{\sqrt{2}}{2} + \tfrac{\sqrt{2}}{2} = \sqrt{2} > 0.$$

So, by the second-derivative test, f has a local maximum at $\frac{\pi}{4}$ and a local minimum at $\frac{5\pi}{4}$. The local maximum value is $f\left(\frac{\pi}{4}\right) = \sqrt{2}$, and the local minimum value is $f\left(\frac{5\pi}{4}\right) = -\sqrt{2}$.

3.7.23. *Find the dimensions of the rectangle of largest area that can be inscribed in a circle of radius r.*

Solution. Let ℓ, w, and A be the length, width, and area respectively of the rectangle. Here is a diagram of the situation:



The diagonal of the rectangle is the diameter of the circle, so the length is $2r$. Thus there is a right triangle with legs ℓ and w, and hypotenuse $2r$. Then the Pythagorean Theorem tells us that $\ell^2 + w^2 = 4r^2$, so $\ell = \sqrt{4r^2 - w^2}$.

The area of the rectangle is $A = w\ell$, and that is what we need to maximize. Substituting $\ell = \sqrt{4r^2 - w^2}$, we get

$$A = w\sqrt{4r^2 - w^2}.$$

Remember that r is a constant, so A is now a function of one variable, w. What is the domain? The width of the rectangle cannot be negative, and it cannot exceed the diameter $2r$ of the circle; so $0 \le w \le 2r$, and the domain is the closed interval $[0, 2r]$.

The derivative (remembering that r is a constant) is:

$$\begin{aligned}\frac{dA}{dw} &= \frac{d}{dw}(w) \cdot \sqrt{4r^2 - w^2} + w \cdot \frac{d}{dw}\left(\sqrt{4r^2 - w^2}\right) \\ &= \sqrt{4r^2 - w^2} + w \cdot \frac{1}{2\sqrt{4r^2 - w^2}} \cdot \frac{d}{dw}\left(4r^2 - w^2\right) \\ &= \sqrt{4r^2 - w^2} + w \cdot \frac{1}{2\sqrt{4r^2 - w^2}} \cdot (-2w) \\ &= \sqrt{4r^2 - w^2} - \frac{w^2}{\sqrt{4r^2 - w^2}}.\end{aligned}$$

There is one number in the domain $[0, 2r]$ where this is undefined, and that is $w = 2r$; but that is an endpoint of the domain, so we won't count it as a critical number. Now we set the derivative to zero:

$$\begin{aligned}\sqrt{4r^2 - w^2} - \frac{w^2}{\sqrt{4r^2 - w^2}} &= 0; \\ \sqrt{4r^2 - w^2} &= \frac{w^2}{\sqrt{4r^2 - w^2}}; \\ \times\sqrt{4r^2 - w^2} \quad & \times \sqrt{4r^2 - w^2} \\ 4r^2 - w^2 &= w^2; \\ w^2 &= 2r^2; \\ w &= \pm\sqrt{2} \cdot r.\end{aligned}$$

We throw out the negative solution because it is not in the domain, so $w = \sqrt{2} \cdot r$ is the only critical number (and it is in the domain).

Now we can use the closed-interval method to find the absolute maximum. We test the critical number and the endpoints in the original function A:

$$A(0) = 0; \qquad A\left(\sqrt{2} \cdot r\right) = 2r^2; \qquad A(2r) = 0.$$

The highest of these values is $\sqrt{2} \cdot r$. Therefore, the absolute maximum area occurs when the width is $w = \sqrt{2} \cdot r$, and the length is $\ell = \sqrt{4r^2 - w^2} = \sqrt{2} \cdot r$. This means the area is maximized when the rectangle is a square!

2.8.4. *The length of a rectangle is increasing at a rate of* 8 *cm/s, and its width is increasing at a rate of* 3 *cm/s. When the length is* 20 *cm and the width is* 10 *cm, how fast is the area of the rectangle increasing?*

Solution. Let ℓ be the length, w the width, and A the area. Then $A = \ell w$. To take the derivative of this, we need to use the product rule:

$$A' = \ell' w + \ell w'.$$

We are given that $\ell = 20$, $w = 10$, $\ell' = 8$, and $w' = 3$. Plugging this in yields

$$A' = 8 \cdot 10 + 20 \cdot 3 = 140.$$

Therefore, the area grows at a rate of 140 cm^2/s.

2.8.5. *A cylindrical tank with radius* 5 *m is being filled with water at a rate of* 3 *m^3/min. How fast is the height of the water increasing?*

Solution. Let h be the height of the water, and let V be the volume of the water. The water is a cylinder whose base is the base of the tank, so it has radius $r = 5$, and so

$$V = \pi r^2 h = 25\pi h.$$

(We can plug in $r = 5$ now because the radius is constant over time, so we don't have to wait until after we take the derivative.) Now the derivative is

$$V' = 25\pi h';$$

and we are given that $V' = 3$, so $3 = 25\pi h'$, and $h' = \dfrac{3}{25\pi} \approx 0.038$ m/s.

2.8.9. *If $x^2 + y^2 + z^2 = 9$, $\frac{dx}{dt} = 5$, and $\frac{dy}{dt} = 4$, find $\frac{dz}{dt}$ when $(x, y, z) = (2, 2, 1)$.*

Solution. We take the relation $x^2 + y^2 + z^2 = 9$ and implicitly differentiate, yielding

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} + 2z\frac{dz}{dt} = 0.$$

Now we just plug in the given information:

$$2 \cdot 2 \cdot 5 + 2 \cdot 2 \cdot 4 + 2 \cdot 1 \cdot \frac{dz}{dt} = 0 \quad \Rightarrow \quad 36 + 2\frac{dz}{dt} = 0 \quad \Rightarrow \quad \frac{dz}{dt} = -18.$$