October 7, 2008 Names:

Lab #2

Proving Stuff With Equivalence Relations

1. An *equivalence relation*, $\sim$, on a set S is a set R of ordered pairs of elements of S such that

   i. $(a, a) \in R$ for all $a \in S$ (reflexive property)

   ii. $(a, b) \in R$ implies $(b, a) \in R$ (symmetric property)

   iii. $(a, b) \in R$ and $(b, c) \in R$ imply $(a, c) \in R$ (transitive property)

   We say a and b are equivalent when $(a, b) \in R$ and write $a \sim b$.

   Define a relation $\sim$ on the set $\mathbb{Z} \times \mathbb{Z}^*$, by $(a, b) \sim (c, d)$ if and only if

   $$ad = bc.$$

   (a) Prove that $\sim$ is an equivalence relation.

   (b) Can this be translated to a relation on $\mathbb{Q}$, the set of rational numbers?

2. For an equivalence relation $\sim$ on a set S and an element $a \in S$, we define

$$[a] = \{x \in S \mid x \sim a\}$$

to be the *equivalence class of* S *containing* a.

(a) Write out what $[(1,2)]$ is for the relation $\sim$ defined above.

(b) Use your translation of the relation onto $\mathbb{Q}$ to describe the equivalence classes of $\sim$ in words.

(c) What do you notice about these equivalence classes? Do they intersect? How many equivalence classes does each rational number belong to? What is the union of all the equivalence classes?

3. A *partition* of a set S is a collection of nonempty disjoint subsets of S whose union is S.

   (a) Draw a picture of a partition of a set.

   (b) Write out the definition of a partition in mathematical symbols.

   (c) Do the equivalence classes above form a partition of $\mathbb{Q}$?

4. It is a fact that the equivalence classes of an equivalence relation on a set S constitute a partition of S, and also that for any partition $\mathcal{P}$ of S, there is an equivalence relation on S whose equivalence classes are the elements of $\mathcal{P}$. We shall prove this:

   (a) For the "forward direction" we suppose we have an equivalence relation, say $\sim$, on S, and show that the equivalence classes partition S. What three properties must we show the set of equivalence classes to have? Write these out in mathematical symbols.

   (b) Prove that the set of equivalence classes has these properties.

(c) For the converse, we must start with a partition, $\mathcal{P}$, of S, and define an equivalence relation whose equivalence classes are exactly the elements of $\mathcal{P}$. Define this relation on S.

(d) Prove that your above relation is actually an equivalence relation.