Instructions: You are encouraged to work out solutions to these problems in groups! Discuss the problems with your classmates and/or your instructor. After doing so, please write up your solutions legibly on a separate sheet (or sheets) of paper (this part should be done on your own). Be sure to use *complete sentences*. Note: Proofs should contain *words*, not just symbols.

1. Let $\mathbb{H}$ be the set $\{1, -1, i, -i, j, -j, k, -k\}$, and define multiplication on $\mathbb{H}$ by the following rules: 1 is the identity, $i^2 = j^2 = k^2 = -1$, $i \cdot j = k$, $j \cdot k = i$, and for any element $a \in \mathbb{H}$, $(-1) \cdot a = a \cdot (-1) = -a$.

   (a) Complete the following Cayley table for $\mathbb{H}$ (you can re-copy it onto your homework, if that is easier).

	1	-1	i	$-i$	j	$-j$	k	$-k$
1								
-1								
i			-1		k			
$-i$								
j					-1		i	
$-j$								
k							-1	
$-k$								

   (b) Is $\mathbb{H}$ Abelian? What happens when you reverse the order of the product of two non-identity (or negative identity) elements? In other words, if $a, b \in \mathbb{H}$ and neither is 1 or -1, how does $b \cdot a$ relate to $a \cdot b$?

   (c) Find a subgroup of order 4 of $\mathbb{H}$, list its elements and use one of the subgroup tests of Chapter 3 to verify that it is, in fact, a subgroup.

2. Recall that we can build the dihedral group of order $2n$, for any n, just as we did for D_4.

   (a) With pictures and words, describe the elements of D_3.

   (b) Identify and describe all subgroups of D_3. Explain how you know these are all the possibilities. (You may use the Cayley table for D_3 which appears at the back of your text book.)

3. The set of all real, upper triangular $n \times n$ matrices is a group under matrix multiplication for any n. Using this fact, prove that the set of real 3×3 matrices of the form $\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$ forms a group under matrix multiplication. This group is called the Heisenberg group. *Hint: This group can be viewed as a subgroup of all* 3×3 *upper triangular matrices.*

4. Prove that the set of orthogonal $n \times n$ matrices,
   $O(n) = \left\{A \in M_n(\mathbb{R}) : A^T A = AA^T = I\right\}$, is a group under matrix multiplication.

Hint: Use facts about the transpose from linear algebra.

5. (Chapter 2, Exercise 14) Let G be a group with the following property: Whenever a, b, and c belong to G and $ab = ca$, then $b = c$. Prove that G is Abelian.