Instructions: You are encouraged to work out solutions to these problems in groups! Discuss the problems with your classmates and/or your instructor. After doing so, please write up your solutions legibly on a separate sheet (or sheets) of paper (this part should be done on your own) and write down the names of the classmates with whom you worked. Be sure to use *complete sentences*. Note: Proofs should contain *words*, not just symbols.

1. We have worked rather extensively with the *group* of quaternions, $\{\pm 1, \pm i, \pm j, \pm k\}$. Here, we'll begin working with the quaternion *ring*.

   (a) Define the ring of quaternions as follows: $\mathbb{H} = \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{R}\}$ (and i, j, and k are the distinct square roots of -1 as in Q). Define addition in the natural way (like vector addition) and for the multiplication, you may assume that all real numbers commute with all imaginary numbers, but the products of the imaginary elements (i, j, k) are defined as the usual products in Q. You may also assume associativity for both $+$ and $\cdot$. Show that $\mathbb{H}$ is a ring.

   (b) Is $\mathbb{H}$ a commutative ring? Does $\mathbb{H}$ have a unity element? If so, what is it? Is $\mathbb{H}$ an integral domain? [1]

   (c) Notice that $\mathbb{C}$ is a subring of $\mathbb{H}$. Which of the properties mentioned in part (b) do $\mathbb{C}$ and $\mathbb{H}$ share? Which properties does only one possess?[2]

2. We mentioned on the second day of class that polynomials (yes, the ones from calculus) actually form a group under addition. Here, we'll consider the *ring* of polynomials with coefficients in various rings.

   (a) Let $\mathbb{R}[x] = \{a_0 + a_1x + \cdots + a_nx^n \mid a_i \in \mathbb{R}\}$. Show that $\mathbb{R}[x]$ is a ring. (You may use the fact that $\mathbb{R}[x]$ is an abelian group under addition of polynomials, but you must show that the relevant properties of multiplication hold.)

   (b) Is $\mathbb{R}[x]$ a commutative ring? Does it have unity? If so, what is it? Is $\mathbb{R}[x]$ an integral domain? What are the units? [3]

   (c) Define $Z_n[x] = \{a_0 + a_1x + \cdots + a_nx^n \mid a_i \in Z_n\}$. To show that $Z_n[x]$ is a ring, you would follow the same steps as in part (a). For what n is $Z_n[x]$ an integral domain? [4]

   (d) What is the characteristic of $Z_n[x]$? (see page 252)

[1] Note: The answers to the questions in 1 (b) should be short. If the answer is no, you should be able to provide a brief illustrative example. You should not have to prove anything here.

[2] Again, I am not asking you to prove anything here.

[3] See the first footnote.

[4] See the second footnote.