Instructions: You are encouraged to work out solutions to these problems in groups! Discuss the problems with your classmates and/or your instructor. After doing so, please write up your solutions legibly on a separate sheet (or sheets) of paper (this part should be done on your own) and write down the names of the classmates with whom you worked. Be sure to use *complete sentences*. Note: Proofs should contain *words*, not just symbols.

1. (Chapter 12, Exercise 19) Let R be a ring. The *center of* R is the set $\{x \in R \mid ax = xa \ \forall a \in R\}$. Prove that the center of a ring is a subring.

2. (Chapter 13, Exercise 40) Show that $Z_7[\sqrt{3}] = \{a + b\sqrt{3} \mid a, b \in Z_7\}$ is a field.

3. (Chapter 14, Exercise 14) Let A and B be ideals of a ring. Prove that $AB \subseteq A \cap B$.

   Note: the product AB is defined in problem 12 as $AB = \{a_1b_1 + a_2b_2 + \cdots + a_nb_n \mid a_i \in A,\ b_i \in B,\ n \in \mathbb{Z}^+\}$. If you find it helpful, you may use the result of problem 12.

4. (Chapter 14, Exercise 16) If A and B are ideals of a commutative ring R with unity and $A + B = R$, show that $A \cap B = AB$.

5. (Chapter 14, Exercise 36) Let R be a ring and let I be an ideal of R. Prove that the factor ring R/I is commutative if and only if $rs - sr \in I$ for all $r, s \in R$.

6. (Chapter 15, Exercise 66) Let $R = \left\{ \begin{bmatrix} a & b \\ b & a \end{bmatrix} \middle| \, a, b \in \mathbb{Z} \right\}$, and let ϕ be the mapping that takes $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$ to $a - b \in \mathbb{Z}$.

   (a) Show that ϕ is a homomorphism.

   (b) Determine the kernel of ϕ.

   (c) Show that $R/\ker\phi$ is isomorphic to $\mathbb{Z}$.

   (d) Is $\ker\phi$ a prime ideal?

   (e) Is $\ker\phi$ a maximal ideal?