Name: ______________________________

Math 31 Take-Home Final

Due December 9, 2009 by 2pm

Instructions: For this exam you may use the assigned course text book, your class notes, and any material on the course Blackboard site. If you are confused about what a question is asking of you, you may consult with your instructor. You may not consult any other source.

This exam is due by 2:00 pm on Wednesday, December 9. It should be returned to Paige's office, 221 Kemeny Hall.

Staple this page to the front of your completed exam before turning it in.

Honor Statement:

I have neither given nor received help on this exam, and all of the answers are my own.

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Signature

Question	Points	Score
1	40	
2	20	
3	15	
4	15	
5	15	
6	35	
Total:	140	

Instructions: For this exam you may use the assigned course text book, your class notes, and any material on the course Blackboard site. If you are confused about what a question is asking of you, I encourage you to come see me. I will be happy to clarify the questions for you. If you have trouble getting started, I will "sell" hints for a small point deduction (only one hint per problem). You may not consult any source other than the ones listed above.

You may cite a result without proof if it appears in either the assigned text, the assigned homework or your class notes and you should specify exactly where it came from (e.g. Problem 3 of HW2, or Thm. 9.3). Even if you cannot solve one part of a problem, you may still use that result in a later part of the problem. **Important:** Show your work and be sure to explain each answer clearly and completely.

1. In this problem, let R be a commutative ring with unity.
   (a) [10 points] Let A and B be proper ideals of R. We say that A and B are comaximal if there is no proper ideal of R containing both A and B. Show that this definition is equivalent to saying that $A + B = R$.
   (b) [10 points] Let A and B be proper ideals of R. Show that the map $\phi : R \to R/A \oplus R/B$ given by $\phi(r) = (r + A, r + B)$ is a ring homomorphism and determine $\ker \phi$.
   (c) [15 points] If the ideals A and B (from part (b)) are comaximal, show that $R/(AB) \cong R/A \oplus R/B$.
   (d) [5 points] The results of parts (b) and (c) form a theorem called the Chinese Remainder Theorem. Restate this theorem in the special case when $R = \mathbb{Z}$. (You should reinterpret the map given in part (b), tell me what friendlier condition means the same as "comaximal" in the case of $\mathbb{Z}$, and describe the ideal AB when the ideals are comaximal, but you do not need to justify these statements).
2. Let D be an integral domain. We say $d \in D$ is a *greatest common divisor* of two nonzero elements $r, s \in D$ if d is a common divisor of r and s (that is, $d \mid r$, $d \mid s$) and whenever $d' \in D$ is another common divisor of r and s, $d' \mid d$.
   (a) [8 points] If d and d' are two greatest common divisors of r and s, how are they related? Explain.
   (b) [12 points] Find a greatest common divisor of $f(x) = 2x^4 + 4x^3 + x + 3$ and $g(x) = 4x^4 + 2x^2 + 4$ in $Z_5[x]$. Hint: These polynomials split over Z_5.
3. [15 points] A ring R with unity is called a *division ring* if every nonzero element in R is a unit. Show that the ring of quaternions, $\mathbb{H} = \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{R};\ i^2 = j^2 = k^2 = -1;\ ij = k = -ji\}$ (defined as in homework), is a division ring. You may assume that $\mathbb{H}$ is a ring with unity.
4. (a) [7 points] Show that $p(x) = x^3 + 9x + 6$ is irreducible in $\mathbb{Q}[x]$.
   (b) [8 points] Let θ be a root of $p(x)$ in some extension of $\mathbb{Q}$. Give a basis for $\mathbb{Q}(\theta)$ and express θ^{-1} in terms of this basis.

5. [15 points] Let F be a field and let α be an element of some algebraic extension K of F. Prove that if $[F(\alpha) : F]$ is odd, then $F(\alpha) = F(\alpha^2)$. Hint: Consider $[F(\alpha) : F(\alpha^2)]$.

6. Let G be a group (written multiplicatively) and let $H, K \subseteq G$ be subgroups. Define $HK = \{hk \mid h \in H, k \in K\}$.

   (a) [10 points] Prove that the following two statements are equivalent:

      1. Every element $x \in HK$ can be written $x = hk$ for a unique element $h \in H$ and a unique element $k \in K$.
      2. $H \cap K = \{e\}$.

   (b) [7 points] Suppose H and K are subgroups of G satisfying the following condition: for every $h \in H$ and $k \in K$, there are elements $h' \in H$ and $k' \in K$ with $hk = k'h$ and $kh = h'k$. Show that HK is a subgroup of G.

   (c) [10 points] Suppose H and K are as in part (b), $G = HK$ and $H \cap K = \{e\}$. Let $\phi : H \oplus K \to G$ by $\phi(h, k) = hk$. Prove that ϕ is an isomorphism.

   (d) [8 points] Let $G = D_3$, $H = \langle R_{120} \rangle$ and $K = \langle F \rangle$. Show that $G = HK$ and $H \cap K = \{R_0\}$. Is $H \oplus K \cong G$? Prove your claim.