X-Hour: Cosets and Factor Groups

10/20/09

Recall: Let G be a group and H a subgroup of G, $H \leq G$.

The left cosets of H in G have the form:

$$aH = \{ah \mid h \in H\} \text{ for } a \in G$$

If G is finite, there are $|G|/|H|$ distinct left cosets.

We say H is a normal subgroup of G

when: $aH = Ha$ for $\forall\, a \in G$.

(we write $H \trianglelefteq G$).

What's special about the cosets of a normal subgroup?

These cosets form a group!

The operation $(aH)(bH) = (ab)H$

Example: Let $G = \mathbb{Z}$ and $H = \langle 4 \rangle = \{\ldots -8, -4, 0, 4, 8, \ldots\}$

Note: $H \trianglelefteq G$ since G is abelian

What are the elements of $G/H = \mathbb{Z}/\langle 4 \rangle$?

$$\left.\begin{array}{l} 0 + \langle 4 \rangle = \{\ldots -8, -4, 0, 4, 8 \ldots\} \\ 1 + \langle 4 \rangle = \{\ldots -7, -3, 1, 5, 9, \ldots\} \\ 2 + \langle 4 \rangle = \{-6, -2, 2, 6, 10 \ldots\} \\ 3 + \langle 4 \rangle = \{-5, -1, 3, 7, 11, \ldots\} \end{array}\right\}$$

these are all the distinct cosets since their union is all of $G = \mathbb{Z}$

So how many distinct cosets are there? 4

Is $\mathbb{Z}/\langle 4 \rangle$ abelian? Yes

Is $\mathbb{Z}/\langle 4 \rangle$ cyclic? Yes. $G/H = \langle \bar{1} \rangle = \langle \bar{3} \rangle$

Write down a Cayley table for the factor group $\mathbb{Z}/\langle 4 \rangle$.

denote by

$\bar{1}$ $1+\langle 4 \rangle$

$\bar{2}$ $2+\langle 4 \rangle$

$\bar{3}$ $3+\langle 4 \rangle$

	$\langle 4 \rangle$	$1+\langle 4 \rangle$	$2+\langle 4 \rangle$	$3+\langle 4 \rangle$
$\langle 4 \rangle$	$\langle 4 \rangle$	$\bar{1}$	$\bar{2}$	$\bar{3}$
$1+\langle 4 \rangle$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\langle 4 \rangle$
$2+\langle 4 \rangle$	$\bar{2}$	$\bar{3}$	$\langle 4 \rangle$	$\bar{1}$
$3+\langle 4 \rangle$	$\bar{3}$	$\langle 4 \rangle$	$\bar{1}$	$\bar{2}$

To what familiar group is the factor group $\mathbb{Z}/\langle 4 \rangle$ isomorphic? $\mathbb{Z}/\langle 4 \rangle \simeq \mathbb{Z}_4$

We know that D_4 is not Abelian. Is the factor group D_4/K Abelian?

Yes (we can see this from the cayley table)

To what familiar group is D_4/K isomorphic? (describe a map, but you don't need to prove this)

$D_4/K \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ *

Find a group homomorphism φ taking D_4 <u>onto</u> the group you identified above (*)

Let $\varphi(R_0) = (0,0)$

$\varphi(R_{90}) = (1,0)$ since φ is supposed to be a homomorphism,

$\varphi(R_{180}) = \varphi(R_{90}) + \varphi(R_{90})$
$= (1,0) + (1,0) = (0,0)$

$\varphi(R_{180}) = (0,0)$

$\varphi(R_{270}) = \varphi(R_{90}) + \varphi(R_{180})$

$\varphi(R_{270}) = (1,0)$

$\varphi(H) = (0,1) \leftarrow$ this we can choose

$\varphi(V) = (0,1) \leftarrow \varphi(V) = \varphi(H) + \varphi(R_{180}) = (0,1) + (0,0)$

$\varphi(D) = (1,1) \leftarrow \varphi(D) = \varphi(HR_{90}) = \varphi(H) + \varphi(R_{90})$
$= (0,1) + (1,0) = (1,1)$

$\varphi(D') = \varphi(D) + \varphi(R_{180})$

$\rightarrow \varphi(D') = (1,1)$

Notice: The elements of the same coset get sent to the same element of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Consider the group D_4 and its subgroup
$K = \{R_0, R_{180}\}$.

Is $K \trianglelefteq D_4$? How do you know?

Yes. $K = Z(D_4)$ (the center of D_4)
and the center is always normal (proved in class)

What are the left cosets of K in D_4?
How many distinct cosets are there?

Since $|D_4| = 8 < \infty$, the number of distinct [left] cosets is
$|D_4|/|K| = 8/2 = 4$.

The distinct cosets are:

$K = R_0K = R_{180}K$ $\quad R_{90}K = R_{270}K = \{R_{90}, R_{270}\}$

$HK = VK = \{H, V\}$ $\quad DK = D'K = \{D, D'\}$

Make a Cayley table for the factor group, D_4/K.

	K	$R_{90}K$	HK	DK
K	K	$R_{90}K$	HK	DK
$R_{90}K$	$R_{90}K$	K	DK	HK
HK	HK	DK	K	$R_{90}K$
DK	DK	HK	$R_{90}K$	K

Let $G = \mathbb{Z}_4 \oplus \mathbb{Z}_2$ and let H be the subgroup of G given by $H = \langle (2,1) \rangle$

What are the elements of G? of H?

$(0,0)$	$(1,0)$	$(2,0)$	$(3,0)$
$(2,1)$	$(3,1)$	$(0,1)$	$(1,1)$

these are the elements of H

Is $H \trianglelefteq G$? How do you know?

Yes, because G is abelian so each of its subgroups is normal.

What are the left cosets of H in G?

How many distinct cosets are there?

Since $|G| = 8 < \infty$, the number of cosets is $|G|/|H| = 8/2 = 4$

$H = (0,0) + H = (2,1) + H = \{(0,0), (2,1)\}$ $\quad (2,0)+H = (0,1)+H = \{(2,0), (0,1)\}$

$(1,0) + H = (3,1) + H = \{(1,0), (3,1)\}$ $\quad (3,0)+H = (1,1)+H = \{(3,0), (1,1)\}$

Make a Cayley table for the factor group, G/H.

	H	$(1,0)+H$	$(2,0)+H$	$(3,0)+H$
H	H	$(1,0)+H$	$(2,0)+H$	$(3,0)+H$
$(1,0)+H$	$(1,0)+H$	$(2,0)+H$	$(3,0)+H$	H
$(2,0)+H$	$(2,0)+H$	$(3,0)+H$	H	$(1,0)+H$
$(3,0)+H$	$(3,0)+H$	H	$(1,0)+H$	$(2,0)+H$

Is the factor group G/H abelian?

Yes, we can see this from the Cayley table.

To what familiar group is G/H isomorphic?
(describe a map, but you don't need to prove this)

$G/H \cong \mathbb{Z}_4$ since G/H is cyclic of order 4.
Since $\langle (1,0)+H \rangle = G/H$, the map $(1,0)+H \mapsto 1$ gives rise to an isomorphism. *

Find a group homomorphism γ taking G <u>onto</u> the group you identified above (*).

Let γ be the map taking:

$\gamma((0,0)) = 0$

$\gamma((1,0)) = 1$

$\gamma((2,0)) = 2$

$\gamma((3,0)) = 3$

$\gamma((2,1)) = 0$

$\gamma((3,1)) = \gamma((2,1)) + \gamma((1,0)) = 0 + 1$

$\gamma((0,1)) = \gamma((2,1)) + \gamma((2,0)) = 0 + 2$

$\gamma((1,1)) = \gamma((2,1)) + \gamma((3,0)) = 0 + 3 = 3.$

We can check to verify that $\gamma((a,b)+(c,d)) = \gamma((a+c, b+d))$ for all $(a,b), (c,d) \in \mathbb{Z}_4 \oplus \mathbb{Z}_2$ ... but it works!

Let $G = \mathbb{Z}_4 \oplus \mathbb{Z}_2$ and let K be the subgroup of G given by $K = \langle (0,1) \rangle$.

What are the elements of G? of K?

The elements of G are:

$(0,0)$ $(1,0)$ $(2,0)$ $(3,0)$

$(0,1)$ $(1,1)$ $(2,1)$ $(3,1)$

these two are the elements of K

Is $K \trianglelefteq G$? How do you know?

Yes, because G is abelian, so all of its subgroups are normal.

What are the left cosets of K in G?

How many distinct cosets are there?

Since $|G| = 8 < \infty$, there are $|G|/|K| = 8/2 = 4$ cosets.

They are:

$K = \{(0,0), (0,1)\}$ $\quad (1,0)+K = (1,1)+K = \{(1,0),(1,1)\}$

$(2,0)+K = (2,1)+K = \{(2,0),(2,1)\}$ $\quad (3,0)+K = (3,1)+K = \{(3,0),(3,1)\}$

Make a Cayley table for the factor group, G/K.

	K	$(1,0)+K$	$(2,0)+K$	$(3,0)+K$
K	K	$(1,0)+K$	$(2,0)+K$	$(3,0)+K$
$(1,0)+K$	$(1,0)+K$	$(2,0)+K$	$(3,0)+K$	K
$(2,0)+K$	$(2,0)+K$	$(3,0)+K$	K	$(1,0)+K$
$(3,0)+K$	$(3,0)+K$	K	$(1,0)+K$	$(2,0)+K$

Is the factor group G/K abelian?

Yes, we can see this from the Cayley table.

To what familiar group is G/K isomorphic?
(describe a map, but you don't need to prove this)

$G/K \cong \mathbb{Z}_4$ (since it is cyclic and of order 4) *

Find a group homomorphism ψ taking G onto the group you identified above (*).

In general, define $\psi : \mathbb{Z}_4 \oplus \mathbb{Z}_2 \to \mathbb{Z}_4$ by

$\psi((a,b)) = a$.

Then ψ is certainly surjective, so let's verify that it's a homomorphism.

Let (a,b) and (c,d) be elements of $\mathbb{Z}_4 \oplus \mathbb{Z}_2$.

Then $\psi((a,b)+(c,d)) = \psi((a+c, b+d))$

$= a+c$

$= \psi((a,b)) + \psi((c,d))$.

In terms of the concrete elements of $\mathbb{Z}_4 \oplus \mathbb{Z}_2$, this means:

$\psi(0,0)=0$ $\psi(1,0)=1$ $\psi(2,0)=2$ $\psi(3,0)=3$

$\psi(0,1)=0$ $\psi(1,1)=1$ $\psi(2,1)=2$ $\psi(3,1)=3$

Notice: Elements in the same coset get sent to the same element of $\mathbb{Z}_4$.

Let $G = \mathbb{Z}_4 \oplus \mathbb{Z}_2$ and let N be the subgroup of G given by $N = \langle (2,0) \rangle$

What are the elements of G? of N?

The elements of G are:

$(0,0)$ $(1,0)$ $(0,1)$ $(1,1)$

$(2,0)$ $(3,0)$ $(2,1)$ $(3,1)$

these are the elements of N

Is $N \trianglelefteq G$? How do you know?

Yes, because G is abelian, so each of its subgroups is normal.

What are the left cosets of N in G?
How many distinct cosets are there?

Since $|G| = 8 < \infty$, there are $|G|/|K| = 8/2 = 4$ cosets

They are:

$N = \{(0,0),(2,0)\} = (0,0)+N = (2,0)+N$

$(1,0)+N = (3,0)+N = \{(1,0),(3,0)\}$ $(0,1)+N = (2,1)+N = \{(0,1),(2,1)\}$

$(1,1)+N = (3,1)+N = \{(1,1),(3,1)\}$

Make a Cayley table for the factor group, G/K.

	N	$(1,0)+N$	$(0,1)+N$	$(1,1)+N$
N	N	$(1,0)+N$	$(0,1)+N$	$(1,1)+N$
$(1,0)+N$	$(1,0)+N$	N	$(1,1)+N$	$(0,1)+N$
$(0,1)+N$	$(0,1)+N$	$(1,1)+N$	N	$(1,0)+N$
$(1,1)+N$	$(1,1)+N$	$(0,1)+N$	$(1,0)+N$	N

Is the factor group G/N abelian?

Yes, we can see this from the Cayley table.

To what familiar group is G/N isomorphic?
(describe a map, but you don't need to prove this)

$G/N \cong Z_2 \oplus Z_2$ (since it is not cyclic and has order 4) *

Define a map $\Psi : G/N \to Z_2 \oplus Z_2$ via

$\Psi((a,b)+N) = (a \bmod 2, b)$

Find a group homomorphism ψ taking G onto the group you identified above (*).

Define $\psi : Z_4 \oplus Z_2 \to Z_2 \oplus Z_2$ by

$\psi((a,b)) = (a \bmod 2, b)$

Note that $\ker \psi = N$