Examples

(1) The splitting field for $x^2 - 2$ over $\mathbb{Q}$ is just $\mathbb{Q}(\sqrt{2})$, since the two roots are $\pm\sqrt{2}$ and $-\sqrt{2} \in \mathbb{Q}(\sqrt{2})$.

(2) The splitting field for $(x^2 - 2)(x^2 - 3)$ is the field $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ generated over $\mathbb{Q}$ by $\sqrt{2}$ and $\sqrt{3}$ since the roots of the polynomial are $\pm\sqrt{2}, \pm\sqrt{3}$. We have already seen that this is an extension of degree 4 over $\mathbb{Q}$ and we have the following diagram of known subfields:

$$\begin{tikzcd}
& \mathbb{Q}(\sqrt{2}, \sqrt{3}) \arrow[dl, dash, "2"'] \arrow[d, dash, "2"] \arrow[dr, dash, "2"] & \\
\mathbb{Q}(\sqrt{2}) \arrow[dr, dash, "2"'] & \mathbb{Q}(\sqrt{6}) \arrow[d, dash, "2"] & \mathbb{Q}(\sqrt{3}) \arrow[dl, dash, "2"] \\
& \mathbb{Q} &
\end{tikzcd}$$

(3) The splitting field of $x^3 - 2$ over $\mathbb{Q}$ is not just $\mathbb{Q}(\sqrt[3]{2})$ since as previously noted the three roots of this polynomial in $\mathbb{C}$ are

$$\sqrt[3]{2}, \quad \sqrt[3]{2}\left(\frac{-1 + i\sqrt{3}}{2}\right), \quad \sqrt[3]{2}\left(\frac{-1 - i\sqrt{3}}{2}\right)$$

and the latter two roots are not elements of $\mathbb{Q}(\sqrt[3]{2})$, since the elements of this field are of the form $a + b\sqrt[3]{2} + c\sqrt[3]{4}$ with rational a, b, c and all such numbers are real.

The splitting field K of this polynomial is obtained by adjoining all three of these roots to $\mathbb{Q}$. Note that since K contains the first two roots above, then it contains their quotient $\dfrac{-1 + \sqrt{-3}}{2}$ hence K contains the element $\sqrt{-3}$. On the other hand, any field containing $\sqrt[3]{2}$ and $\sqrt{-3}$ contains all three of the roots above. It follows that

$$K = \mathbb{Q}(\sqrt[3]{2}, \sqrt{-3})$$

is the splitting field of $x^3 - 2$ over $\mathbb{Q}$. Since $\sqrt{-3}$ satisfies the equation $x^2 + 3 = 0$, the degree of this extension over $\mathbb{Q}(\sqrt[3]{2})$ is at most 2, hence must be 2 since we observed above that $\mathbb{Q}(\sqrt[3]{2})$ is not the splitting field. It follows that

$$[\mathbb{Q}(\sqrt[3]{2}, \sqrt{-3}) : \mathbb{Q}] = 6.$$

Note that we could have proceeded slightly differently at the end by noting that $\mathbb{Q}(\sqrt{-3})$ is a subfield of K, so that the index $[\mathbb{Q}(\sqrt{-3}) : \mathbb{Q}] = 2$ divides $[K : \mathbb{Q}]$. Since this extension degree is also divisible by 3 (because $\mathbb{Q}(\sqrt[3]{2}) \subset K$), the degree is divisible by 6, hence must be 6.

This gives us the diagram of known subfields:

$$\begin{tikzcd}
& \mathbb{Q}(\sqrt[3]{2}, \sqrt{-3}) \arrow[dl, dash] \arrow[r, dash] \arrow[rr, dash] \arrow[rrr, dash] & \mathbb{Q}(\theta_1) \arrow[dl, dash] & \mathbb{Q}(\theta_2) \arrow[dll, dash] & \mathbb{Q}(\theta_3) \arrow[dlll, dash] \\
\mathbb{Q}(\sqrt{-3}) \arrow[r, dash] & \mathbb{Q} & & &
\end{tikzcd}$$

where

$$\theta_1 = \sqrt[3]{2}, \quad \theta_2 = \sqrt[3]{2}\left(\frac{-1 + i\sqrt{3}}{2}\right), \quad \theta_3 = \sqrt[3]{2}\left(\frac{-1 - i\sqrt{3}}{2}\right).$$

(4) One must be careful in computing splitting fields. The splitting field for the polynomial $x^4 + 4$ over $\mathbb{Q}$ is smaller than one might at first suspect. In fact this polynomial factors over $\mathbb{Q}$:

$$\begin{aligned} x^4 + 4 &= x^4 + 4x^2 + 4 - 4x^2 = (x^2 + 2)^2 - 4x^2 \\ &= (x^2 + 2x + 2)(x^2 - 2x + 2) \end{aligned}$$

where these two factors are irreducible (Eisenstein again). Solving for the roots of the two factors by the quadratic formula, we find the four roots

$$\pm 1 \pm i$$

so that the splitting field of this polynomial is just the field $\mathbb{Q}(i)$, an extension of degree 2 of $\mathbb{Q}$.

Examples of splitting fields borrowed from Dummit + Foote's *Abstract Algebra*, 3rd Ed. (p. 537-538)