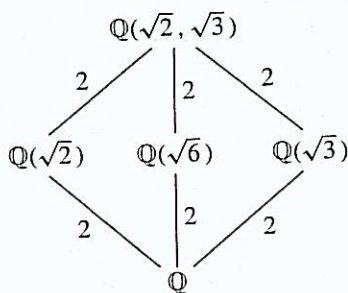


### Examples

- (1) The splitting field for  $x^2 - 2$  over  $\mathbb{Q}$  is just  $\mathbb{Q}(\sqrt{2})$ , since the two roots are  $\pm\sqrt{2}$  and  $-\sqrt{2} \in \mathbb{Q}(\sqrt{2})$ .
- (2) The splitting field for  $(x^2 - 2)(x^2 - 3)$  is the field  $\mathbb{Q}(\sqrt{2}, \sqrt{3})$  generated over  $\mathbb{Q}$  by  $\sqrt{2}$  and  $\sqrt{3}$  since the roots of the polynomial are  $\pm\sqrt{2}, \pm\sqrt{3}$ . We have already seen that this is an extension of degree 4 over  $\mathbb{Q}$  and we have the following diagram of known subfields:



- (3) The splitting field of  $x^3 - 2$  over  $\mathbb{Q}$  is not just  $\mathbb{Q}(\sqrt[3]{2})$  since as previously noted the three roots of this polynomial in  $\mathbb{C}$  are

$$\sqrt[3]{2}, \quad \sqrt[3]{2} \left( \frac{-1 + i\sqrt{3}}{2} \right), \quad \sqrt[3]{2} \left( \frac{-1 - i\sqrt{3}}{2} \right)$$

and the latter two roots are not elements of  $\mathbb{Q}(\sqrt[3]{2})$ , since the elements of this field are of the form  $a + b\sqrt[3]{2} + c\sqrt[3]{4}$  with rational  $a, b, c$  and all such numbers are real.

The splitting field  $K$  of this polynomial is obtained by adjoining all three of these roots to  $\mathbb{Q}$ . Note that since  $K$  contains the first two roots above, then it contains their quotient  $\frac{-1 + \sqrt{-3}}{2}$  hence  $K$  contains the element  $\sqrt{-3}$ . On the other hand, any field containing  $\sqrt[3]{2}$  and  $\sqrt{-3}$  contains all three of the roots above. It follows that

$$K = \mathbb{Q}(\sqrt[3]{2}, \sqrt{-3})$$

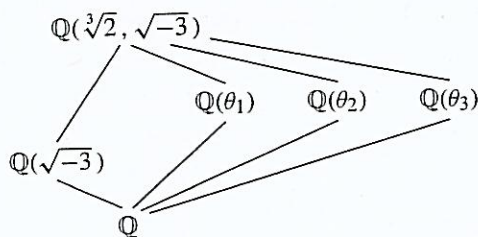
is the splitting field of  $x^3 - 2$  over  $\mathbb{Q}$ . Since  $\sqrt{-3}$  satisfies the equation  $x^2 + 3 = 0$ , the degree of this extension over  $\mathbb{Q}(\sqrt[3]{2})$  is at most 2, hence must be 2 since we observed above that  $\mathbb{Q}(\sqrt[3]{2})$  is not the splitting field. It follows that

$$[\mathbb{Q}(\sqrt[3]{2}, \sqrt{-3}) : \mathbb{Q}] = 6.$$

Note that we could have proceeded slightly differently at the end by noting that  $\mathbb{Q}(\sqrt{-3})$  is a subfield of  $K$ , so that the index  $[\mathbb{Q}(\sqrt{-3}) : \mathbb{Q}] = 2$  divides  $[K : \mathbb{Q}]$ .

Since this extension degree is also divisible by 3 (because  $\mathbb{Q}(\sqrt[3]{2}) \subset K$ ), the degree is divisible by 6, hence must be 6.

This gives us the diagram of known subfields:



where

$$\theta_1 = \sqrt[3]{2}, \quad \theta_2 = \sqrt[3]{2} \left( \frac{-1 + i\sqrt{3}}{2} \right), \quad \theta_3 = \sqrt[3]{2} \left( \frac{-1 - i\sqrt{3}}{2} \right).$$

- (4) One must be careful in computing splitting fields. The splitting field for the polynomial  $x^4 + 4$  over  $\mathbb{Q}$  is smaller than one might at first suspect. In fact this polynomial factors over  $\mathbb{Q}$ :

$$\begin{aligned} x^4 + 4 &= x^4 + 4x^2 + 4 - 4x^2 = (x^2 + 2)^2 - 4x^2 \\ &= (x^2 + 2x + 2)(x^2 - 2x + 2) \end{aligned}$$

where these two factors are irreducible (Eisenstein again). Solving for the roots of the two factors by the quadratic formula, we find the four roots

$$\pm 1 \pm i$$

so that the splitting field of this polynomial is just the field  $\mathbb{Q}(i)$ , an extension of degree 2 of  $\mathbb{Q}$ .

Examples of splitting fields borrowed from Dummit + Foote's  
 Abstract Algebra, 3<sup>rd</sup> Ed. (p. 537-538)