NAME: GRADE:

Math 31

Final - Practice Exam

Grading

- Solutions must be **justified with full sentences**.
- The clarity of your explanations will enter into the appreciation of your work.

Problem	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	Total
Points	5	6	6	5	6	5	5	6	6	50
Score										

1. *(5 points)* Let $(A, +, \cdot)$ be an integral domain and set

$$S := \{(a, b), a, b \in A, b \neq 0\} = A \times A \backslash \{0\}.$$

Show that for $(a, b), (c, d) \in S$ the relation

$$(a, b) \sim (c, d) \Leftrightarrow ad = bc$$

is an equivalence relation.

2. *(6 points)* Let $a = (1\,3\,4)$ and $b = (2\,5\,6)$ be two elements in $(S_6, \circ)$.

a. Determine the order $\operatorname{ord}(a \circ b)$ of $a \circ b$.

b. How many elements does the subgroup $H = \langle a, b \rangle$ generated by a and b have? Justify your answer.

c. Is H an abelian subgroup? Justify your answer.

3. *(6 points)* Let

$$\mathcal{T} = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}, a, b, c \in \mathbb{R} \right\}$$

be the set of upper triangular matrices and $(\mathcal{T}, +, \cdot)$ be the corresponding ring with matrix addition and multiplication.

a. Show that the function $f : \mathcal{T} \to \mathbb{R} \times \mathbb{R}$ defined by

$$f \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} := (a, c)$$

is a ring homomorphism.

b. Determine the kernel $\ker(f)$ and the image $f(\mathcal{T})$ of f. Simplify your answer as much as possible.

4. *(5 points)* Consider the cyclic group $(\mathbb{Z}_n, +_n)$. For r in $\{1, 2, ..., n\}$ show that:
The subgroup $\langle r \rangle$ of $\mathbb{Z}_n$ generated by r is equal to $\mathbb{Z}_n$ **if and only if** r and n are relatively prime.

Note: r and n are *relatively prime* if 1 is the only common factor of r and n.

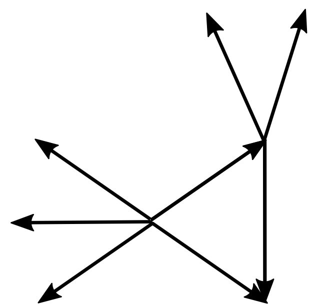
5. *(6 points)* Consider the product ring $\mathbb{Z} \times \mathbb{Z}$. Let $J \lhd \mathbb{Z} \times \mathbb{Z}$ be the ideal

$$J = \langle 3 \rangle \times \mathbb{Z} = \{(3n, m), n, m \in \mathbb{Z}\}.$$

a. Determine whether or not J is a prime ideal in $\mathbb{Z} \times \mathbb{Z}$.

b. Determine whether or not J is a maximal ideal in $\mathbb{Z} \times \mathbb{Z}$.

6. *(5 points)* Consider the following graph Γ:



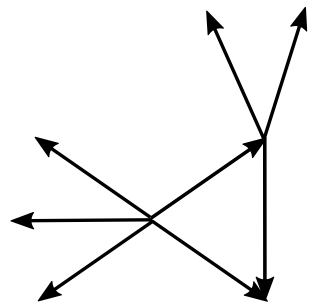
a. Show that $(\mathrm{Aut}(\Gamma), \circ)$, the automorphism group of Γ, is (isomorphic to) a subgroup of $(S_5, \circ)$.

b. Is $(\mathrm{Aut}(\Gamma), \circ)$ abelian? Justify your answer.

c. How many elements does $(\mathrm{Aut}(\Gamma), \circ)$ have? Justify your answer.

d. Determine $(\mathrm{Aut}(\Gamma), \circ)$.

7. *(5 points)* Draw a Cayleygraph of $(\mathrm{Aut}(\Gamma), \circ)$ where Γ is the graph from **problem 6**.



Hint: Three elements are sufficient to generate $(\mathrm{Aut}(\Gamma), \circ)$. Which part of the graph is repeated?

8. *(6 points)* Let $(A,+,\cdot)$ be an integral domain. Recall that $(A,+,\cdot)$ can be extended to a field A^* of "fractions" whose elements are equivalence classes in $A \times (A\backslash\{0\})$. For $(a,b),(r,s) \in A \times (A\backslash\{0\})$ we have

$$(a,b) \sim (r,s) \Leftrightarrow as = br.$$

Multiplication and addition are defined by

$$[a,b] + [c,d] = [ad+bc, bd] \text{ and } [a,b]\cdot[c,d] = [ac, bd].$$

a. If $[a,b] = [r,s]$ and $[c,d] = [t,u]$, prove that $[a,b]+[c,d] = [r,s]+[t,u]$.

b. Show that A^* and A have the same characteristic, i.e. $char(A) = char(A^*)$.

9. *(6 points)* Let $(A, +, \cdot)$ be a ring. Let $B \leq A$ be a subring and $J \lhd A$ an ideal in A.

a. Show that $B + J = \{b + j, \;\text{ where } b \in B \;\text{ and }\; j \in J\}$ is a subring.

b. Show that $B \cap J$ is an ideal in the subring $(B, +, \cdot)$.

c. You may assume that $f : B \to (B + J)/J, b \mapsto f(b) := (b + 0) + J = b + J$ is a ring homomorphism. Determine its kernel $\ker(f)$ and its image $f(B)$. Simplify your answer as much as possible.

d. Apply the fundamental homomorphism theorem for rings to $f : B \to (B + J)/J$.