Math 31: Abstract Algebra
Fall 2017 - Homework 7

Return date: Wednesday 11/01/17

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keywords: *homomorphism theorem, graph automorphisms*

Instructions: Write your answers neatly and clearly on straight-edged paper, use complete sentences and label any diagrams. Please show your work; no credit is given for solutions without work or justification.

exercise 1. *(4 points)* Let $f : A \to C$ and $g : B \to D$ be two functions.
Let $f \times g : A \times B \to C \times D$ be function defined by

$$(f \times g)(a, b) := (f(a), g(b)) \quad \text{for all} \quad (a, b) \in A \times B.$$

Show that $f \times g$ bijective $\Leftrightarrow$ f and g bijective.

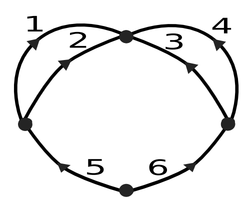
exercise 2. *(6 points)* Let $(G, \cdot)$ and $(H, \cdot)$ be two groups. Let furthermore $N \lhd G$ be a normal subgroup of G and $M \lhd H$ be a normal subgroup of H.

a) Show that the function $f : G \times H \to (G/N) \times (H/M), (a, b) \mapsto f(a, b) := (Na, Mb)$ is a surjective homomorphism.

b) Find the kernel $\ker(f)$ of f.

c) Use the homomorphism theorem to conclude that $(G \times H)/(N \times M) \simeq (G/N) \times (H/M)$.

exercise 3. *(7 points)* Let Γ be the following graph with edges $E = \{1, 2, 3, 4, 5, 6\}$.



a) Show that $\mathrm{Aut}(\Gamma)$ is (isomorphic to) a subgroup of $(S_4, \circ)$.

b) Show that $\mathrm{Aut}(\Gamma)$ contains an element r of order 4 and find $\#\,\mathrm{Aut}(\Gamma)$.

c) Show that $\mathrm{Aut}(\Gamma)$ is a non-abelian group.

d) Draw the Cayley graph of $(\mathrm{Aut}(\Gamma), \circ)$ with generators r and s, where s is an element of order 2. Then look up the subgroups of S_4 and decide to which group $\mathrm{Aut}(\Gamma)$ is isomorphic.

exercise 4. *(3 points)* Let $\Gamma := \Gamma(\mathbb{Z}_5, \{1\})$ be the Cayley graph of $\mathbb{Z}_5$ generated by $\{1\}$. Show that $\mathrm{Aut}(\Gamma) \simeq (\mathbb{Z}_5, +_5)$.
Note: In general $\mathrm{Aut}(\Gamma(\mathbb{Z}_n, \{1\})) \simeq (\mathbb{Z}_n, +_n)$.

Math 31 – Abstract Algebra
Fall 17 – Homework 7

exercise 1: $f: A \to C$, $g: B \to D$ functions.

$$f \times g : A \times B \to C \times D$$
$$(a,b) \mapsto (f \times g)(a,b) := (f(a), g(b))$$

To show: $f \times g$ bijective $\iff$ f and g bijective

proof: $f \times g$ bijective $\iff$ $f \times g$ has an inverse function $(f \times g)^{-1} : C \times D \to A \times B$, such that

1.) $(f \times g) \circ (f \times g)^{-1} = id_{C \times D}$, $(f \times g)^{-1} \circ f \times g = id_{A \times B}$

Let $(f \times g)^{-1} = (\tilde{f}, \tilde{g})$. Then 1.) reads:

$\forall c,d \in C \times D$: $\quad (f \times g)(\tilde{f}(c), \tilde{g}(d)) = (f \circ \tilde{f}(c), g \circ \tilde{g}(d)) = (c,d)$

$\forall a,b \in A \times B$: $\quad (\tilde{f} \times \tilde{g})(f(a), f(b)) = (\tilde{f} \circ f(a), \tilde{g} \circ g(b)) = (a,b)$

$$\iff \begin{pmatrix} f \circ \tilde{f} = id_C & \text{and} & \tilde{f} \circ f = id_A \\ g \circ \tilde{g} = id_D & \text{and} & \tilde{g} \circ g = id_B \end{pmatrix}$$

Then g and f have inverse functions $\tilde{g}$ and $\tilde{f}$ and therefore are bijective

Conversely, if g and f have inverse functions g^{-1} and f^{-1} the inverse steps yield:

$$(f \times g) \circ (f^{-1} \times g^{-1}) = id_{C \times D}$$
$$(f^{-1} \times g^{-1}) \circ f \times g = id_{A \times B}$$

Hence $(f \times g)^{-1} = f^{-1} \times g^{-1}$ and $f \times g$ is bijective.

exercise 2: $(G,\cdot), (H,\cdot)$ groups. $N \triangleleft G$ normal, $M \triangleleft H$ normal.

a) Show that $f: G \times H \to (G/N) \times (H/M)$
$(a,b) \mapsto f(a,b) := (Na, Mb)$

is a surjective homomorphism.

proof: 1.) f homomorphism: $\forall (a,b), (c,d) \in G \times H$

$f((a,b)\cdot(c,d)) = f((ac,bd)) = (Nac, Mbd)$
$\hphantom{f((a,b)\cdot(c,d)) = f((ac,bd))} =$
$f(a,b) \cdot f(c,d) = (Na, Mb)\cdot(Nc, Md)$

This is a homomorphism, as the operation is done entrywise and $Na\,Nc = Nac$, $Mb\,Md = Mbd$.

2.) f surjective: $\forall (Na, Nb) \in G/N \times H/M$:
$f(a,b) = (Na, Nb)$
hence f is surjective.

b) find the kernel ker(f):

$\ker(f) = \{(a,b) \in G \times H, f(a,b) = (Na, Mb) = (N, M)\}$

We know $(Na, Mb) = (N, M) \iff Na = N$ and $Mb = M$
$\iff a \in N$ and $b \in M$

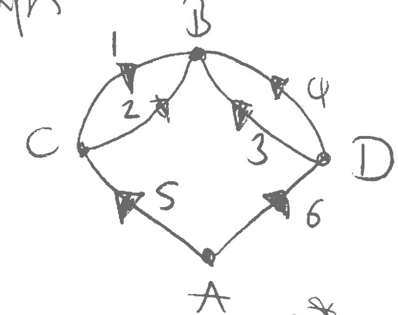
This follows from Chap. 15, Th. 5 (book)

Hence $\ker(f) = N \times M$

c) By the homomorphism theorem:

$G \times H / N \times M = G \times H / \ker f \cong f(G \times H) = G/N \times H/M$

exercise 3: Γ graph



a) Show that $Aut(\Gamma) \simeq G^*$, $G^* \leq S_4$

Let $f \in Aut(\Gamma)$. 1.) We know: If $v \in V$ is a vertex then $f_V(val(v)) = val(v)$, hence

$f_V(A) = A$ and $f_V(B) = B$

2.) Looking at the edge 1 we have two options

a) 1 stays with C or b) 1 goes to D

In both cases the image of 5 and 6 under f is already defined.

3.) As $f_V(1)$ defines $f_V(5)$ and $f_V(6)$ we see that f is already completely defined by

$f_E(\{1,2,3,4\}) = \{1,2,3,4\}$

hence we consider f as an element of $(S_4, \circ)$

Hence $f \in Aut(\Gamma) \simeq G^* \leq S_4$ as $Aut(\Gamma)$ is a group.

b) We write $f \in S_4$ as a permutation of the edges $\{1,2,3,4\}$

To find all elements we distinguish 2 cases

$\# Aut(\Gamma) = 8$

a) $f_V(C) = C$: The possibilities are:

id, $f_1 = (12)$, $f_2 = (34)$, $f_3 = (12) \circ (34)$

b) $f_V(C) = D$: The possibilities are:

$f_4 = (14) \circ (23)$, $f_5 = (13) \circ (24)$, $f_6 = (1423)$, $f_7 = (1324)$

These are all possibilities as 1 and 2 and 3 and 4 must be together on the same side.

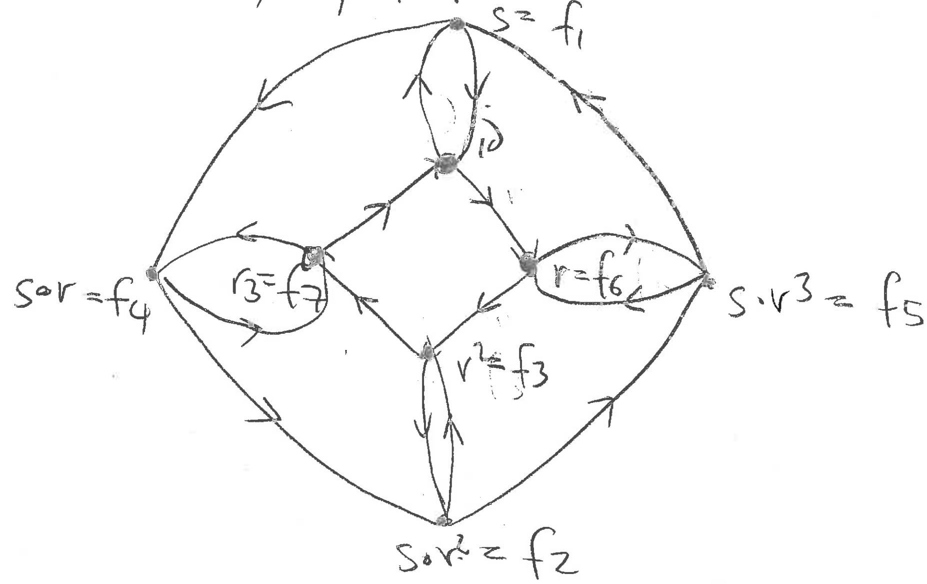
The element of order 4 is $r = (1423) = f_6$

c) Take $s = (12) = f_1$. Then

$$(12)\circ(1423) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} = (14)\circ(23) = f_4$$
$$\neq$$
$$(1423)\circ(12) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} = (13)\circ(24) = f_5$$

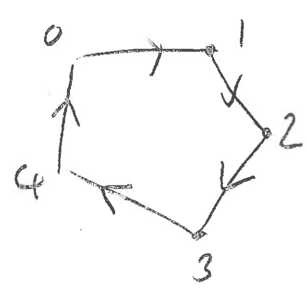
Hence $s\circ r \neq r\circ s$ and $\mathrm{Aut}(\Gamma)$ is not abelian.

d) Draw the Cayley graph $\tilde{\Gamma}(\mathrm{Aut}(\Gamma), \{r,s\})$



This group is isomorphic to D_8, the symmetry group of the square as S_4 contains only one group of order 8.

exercise 4: $\Gamma = \Gamma(\mathbb{Z}_5, \{1\})$



To show: $\mathrm{Aut}(\Gamma) \cong \mathbb{Z}_5$

1.) We know that $\mathbb{Z}_5 \cong G^* \leq \mathrm{Aut}(\Gamma)$ by the lecture (Cayley graphs II)

2.) Let $f: \Gamma \to \Gamma$ be a graph automorphism. Then $f_V(0)$ already determines f!

$$f_V(0) = k \underset{\text{morphism}}{\overset{f \text{ graph}}{\Longrightarrow}} f_V(1) = k +_5 1 \Rightarrow f_V(2) = k +_5 2$$

$$\cdots \Rightarrow f_V(k+5) = k$$

This is true as the graph morphism must respect endpoints and directions of arrows.

There are only 5 possibilities for $f_V(0)$.

Hence $\mathrm{Aut}(I) \simeq (\mathbb{Z}_5, +_5)$

The same arguments apply to $(\mathbb{Z}_n, +_n)$ and we get

$$\mathrm{Aut}(I(\mathbb{Z}_n, \{1\})) \simeq (\mathbb{Z}_n, +_n)$$