Math 31: Abstract Algebra
Fall 2017 - Quiz 1

Date: 09/28/17

Test your knowledge

True false questions *(1 point each)*

1. $+_4$ is an operation on the set $\mathbb{Z}_2 = \{0, 1\}$. ◯ True ◯ False
   False. This operation is not closed, as $1 +_4 1 = 1 + 1 \mod 4 = 2$, but 2 is not in $\mathbb{Z}_2$.

2. Let $*$ be an operation on a set A. If $(A, *)$ has a neutral element e, then e is unique. ◯ True ◯ False
   True. Let e_1, e_2 be two neutral elements in A, then $e_1 = e_1 * e_2 = e_2$.

3. Let $(G, \cdot)$ be a group and $a, b \in G$. Then $(ab)^2 = a^2b^2$. ◯ True ◯ False
   False. In general $(ab)^2 = abab \neq aabb = a^2b^2$.

4. Let $(G, \cdot)$ be a group and H and K subgroups of G. Then $H \cup K$ is a subgroup of G. ◯ True ◯ False
   False. Counterexample: $\langle 2 \rangle$ and $\langle 3 \rangle$ are subgroups of $(\mathbb{Z}, +)$, but $\langle 2 \rangle \cup \langle 3 \rangle$ is not a subgroup, as $3 - 2 = 1 \notin \langle 2 \rangle \cup \langle 3 \rangle$.

5. The set $H = \{f : \mathbb{R} \to \mathbb{R} \mid f(x) \geq 0 \text{ for all } x \in \mathbb{R}\}$ is a subgroup of $(\mathcal{F}(\mathbb{R}), +)$. ◯ True ◯ False
   False. The function $g : \mathbb{R} \to \mathbb{R}, x \mapsto g(x) = 1$ is in H, but $-g$ is not in H.

6. Let $(G, \cdot)$ be a group, $a, b \in G$ fixed and $f : G \to G, x \mapsto f(x) = axb$. Then f is bijective. ◯ True ◯ False
   True. The inverse function f^{-1} is given by $f^{-1}(x) := a^{-1}xb^{-1}$ for all $x \in G$.

7. Let $(G, \cdot)$ be a group. $S \subset G$, such that $\#S = n$ and $\langle S \rangle = G$. Then G has only finitely many elements. ◯ True ◯ False
   False. $(\mathbb{Z}, +) = \langle 1 \rangle$.

8. If G and H are groups, such that $\#G = n$ and $\#H = m$. Then $\#(G \times H) = n + m$. ◯ True ◯ False
   False. It follows from the defintion of the cartesian product that $\#(G \times H) = n \cdot m$.

9. $(\mathcal{F}(\mathbb{R}), \cdot)$ is a group with neutral element $1 : \mathbb{R} \to \mathbb{R}, x \mapsto 1(x) = 1$. ◯ True ◯ False
   False. The inverse function for a function f would be $\frac{1}{f}$, but if $f(x) = 0$ this function is not defined for x.

10. $(\mathbb{Q}, +)$ is isomorphic to $(\mathbb{Z}, +)$. **Hint:** If $F : \mathbb{Q} \to \mathbb{Z}$ is an isomorphism. If $F(q) = 1$, what is $F(\frac{q}{2})$? ◯ True ◯ False
    False. We get $F(\frac{q}{2}) + F(\frac{q}{2}) = F(\frac{q}{2} + \frac{q}{2}) = F(q) = 1$, hence $2F(\frac{q}{2}) = 1$ or $F(\frac{q}{2}) = \frac{1}{2}$. But $\frac{1}{2}$ is not in $\mathbb{Z}$. Hence there can not be such an isomorphism.

Long answer questions

question 1 *(5 points)* Let $G = \{e, a, b, c\}$ be a set of four elements, where e denotes the neutral element. Using an operation table, find all possible groups with four elements, where each element is its own inverse.

Solution: Denote by $*$ the operation. As $(G, *)$ has to be a group, we have as each element is its own inverse:

*	e	a	b	c
e	e	a	b	c
a	a	e		
b	b		e	
c	c			e

The rule is that each element occurs **exactly once** in each column and each row. We look at $a * b$. By this rule (look at the corresponding row **and** column) the only option is $a * b = c$. By the same argument $b * a = c$. It then follows that $a * c = c * a = b$. Finally it follows that $b * c = c * b = a$. Hence the table is

*	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

This means that this group is isomorphic to $(\mathbb{Z}_2 \times \mathbb{Z}_2, +_2 \times +_2)$.

question 2 *(5 points)* Let $(G, \cdot)$ be a group and $H = \langle\{a, b\}\rangle$ be the subgroup generated by the elements a and b, which satisfy the equations

$$a^2 = e \quad , \quad b^3 = e \quad , \quad ab = ba.$$

a) Show that H is an abelian group.
Proof: As $ab = ba$ this implies

$$ab = ba \quad , \quad ab^{-1} = b^{-1}a \quad , \quad a^{-1}b = ba^{-1} \quad , \quad a^{-1}b^{-1} = b^{-1}a^{-1}.$$

Let h_1 and h_2 be two elements in H. Then h_1 is sequence of letters in the set $\{a, b, a^{-1}, b^{-1}\}$ and so is h_2. The above conditions imply that we can change the order of those letters arbitrarily. Especially that implies that

$$h_1 \cdot h_2 = h_2 \cdot h_1.$$

This implies that H is abelian.

b) How many different elements can H contain at most?
Solution: As $a^2 = e$, we have that $a^{-1} = a$. As $b^3 = e$, we have that $b^{-1} = b^2$.
We get that

$$H = \{e, a, b, b^2, ab, ab^2\}.$$

All other words in $\{a, b, b^2\}$ can be reduced to the elements above as H is abelian.