Math 31: Exam 2 Practice

Date: 10/24/19

Test your knowledge

True/false questions

1. The family of sets $\{A_r \mid r \in \mathbb{R}\}$ defined by $A_r = \{(x, y) \mid x^2 + y^2 = r^2\}$ forms a partition of $\mathbb{R} \times \mathbb{R}$. True

2. Let $\langle G, \cdot \rangle$ be a group and H a subgroup. If $a \notin Hb$, then $Ha \neq Hb$. True

3. Let $\langle G, \cdot \rangle$ be a group and H a subgroup. For a fixed $a \in G$ the function $f : aH \to Ha$ defined by $f(ah) = ha$ is a bijective function. True

4. Let $\langle G, \cdot \rangle$ be a group and H a subgroup. Then for any $a \in G$ we have aHa^{-1} is a subgroup of G, where $aHa^{-1} = \{aha^{-1} \mid h \in H\}$. True

5. Let G be a finite group, and let H and K be subgroups of G with $H \subset K$. Then $(G : H) = (G : K)(K : H)$. True

6. Let G be a group. The function $f : G \to G$ defined by $f(x) = x^3$ is a homomorphism. False

7. $\langle \mathcal{F}(\mathbb{R}), +, \cdot \rangle$, where $\cdot$ is function multiplication ($(f \cdot g)(x) := f(x)g(x)$), is a ring. True

8. If $\langle A, +, \cdot \rangle$ is a commutative ring and $b \in A$ a divisor of zero. Then $n \bullet b$ is either zero or a divisor of zero. True

9. If n is not a prime then $\langle \mathbb{Z}_n, +_n, \cdot_n \rangle$ is not an integral domain. True

Note: We have not covered the following material. These are for practice after Friday's class, and solutions will be posted online.

10. If A is a ring and $I \lhd A$ and $J \lhd A$ are ideals then $I \cap J$ is an ideal. True

11. In $\mathbb{Z}_5 \times \mathbb{Z}_5$ the set $B = \{(2n, 2n), n \in \mathbb{Z}_5\}$ is a subring. True

12. In $\mathbb{Z}_5 \times \mathbb{Z}_5$ the set $B = \{(2n, 2n), n \in \mathbb{Z}_5\}$ is an ideal. False

13. Let $\alpha : \langle \mathcal{F}(\mathbb{R}), +, \cdot \rangle \to \langle \mathbb{R}, +, \cdot \rangle$ be the map defined by $\alpha(f) := f(3) - f(0)$. Then α is a ring homomorphism. False

Long answer questions

Question 1 Prove that $m \sim n$ iff $|m| = |n|$ is an equivalence relation on $\mathbb{Z}$. Then, describe the partition associated with that equivalence relation.

Reflexive: Since $|m| = |m|$, we have that $m \sim m$.
Symmetric: If $m \sim n$, then $|m| = |n|$. This implies that $|n| = |m|$ and so $n \sim m$.
Transitive: If $m \sim n$ and $n \sim k$, then $|m| = |n| = |k|$ and so $m \sim k$.
So $\sim$ is an equivalence relation on $\mathbb{Z}$. The associated partition is

$$\{0\}, \{\pm 1\}, \{\pm 2\}, \{\pm 3\}, \ldots$$

Question 2 Write down the cosets of the subgroup $\langle \frac{1}{3} \rangle$ generated by $\frac{1}{3}$.

a) Find all of the cosets of the subgroup $\langle \frac{1}{3} \rangle \leq \langle \mathbb{R}^*, \cdot \rangle$, where $\mathbb{R}^*$ is the set of nonzero real numbers. What is the index of $\langle \frac{1}{3} \rangle$ in $\langle \mathbb{R}^*, \cdot \rangle$?

   Note that $\langle 1/3 \rangle = \{\ldots, 1/9, 1/3, 1, 3, 9, \ldots\} = \langle 3 \rangle$. So any coset would be of the form

$$\langle 1/3 \rangle r = \{r 3^n \mid n \in \mathbb{Z}\}$$

   where $0 < r \leq 1$. The index is infinite.

b) Find all of the cosets of the subgroup $\langle \frac{1}{3} \rangle \leq \langle \mathbb{R}, + \rangle$. What is the index of $\langle \frac{1}{3} \rangle$ in $\langle \mathbb{R}, + \rangle$?

   Note that $\langle 1/3 \rangle = \{\ldots, -1, -2/3, -1/3, 0, 1/3, 2/3, 1, \ldots\} = \{k/3 \mid k \in \mathbb{Z}\}$.
   So any coset would be of the form

$$\langle 1/3 \rangle + r = \{r + k/3 \mid k \in \mathbb{Z}\}$$

   where $r \in \mathbb{R}$. The index is infinite.

Question 3 Find all the normal subgroups (a) of S_3 and (b) of D_4.

In S_3: $\{\epsilon\}, \{\epsilon, (123), (132)\}, S_3$

In D_4: $\{\epsilon\}, \langle r \rangle, \langle r^2 \rangle, \langle r^2, f \rangle, \langle r^2, rf \rangle$, and D_4.
Here r is a rotation by 90 degrees cw and f is a reflection.

Question 4 Let $K \lhd G$ and $H \lhd K$. Show that if G/H is an abelian group, then both G/K and K/H are abelian groups.