Math 31: Exam 1 Practice

Date: 10/03/19

Test your knowledge

True/false questions

1. $+_4$ is an operation on the set $\mathbb{Z}_2 = \{0, 1\}$. ◯ True ◯ False

2. Let $*$ be an operation on a set A. If $(A, *)$ has a neutral element e, then e is unique.
◯ True ◯ False

3. Let $\langle G, \cdot \rangle$ be a group and $a, b \in G$. If a and b commute, then a^2 commutes with b^2.
◯ True ◯ False

4. Let $\langle G, \cdot \rangle$ be a group and H and K subgroups of G. Then $H \cup K$ is a subgroup of G.
◯ True ◯ False

5. The set $H = \{f : \mathbb{R} \to \mathbb{R} \mid f(x) \geq 0 \text{ for all } x \in \mathbb{R}\}$ is a subgroup of $(\mathcal{F}(\mathbb{R}), +)$.
◯ True ◯ False

6. Let $(G, \cdot)$ be a group, $a, b \in G$ fixed and $f : G \to G$ be the function defined by $f(x) = axb$. Then f is bijective. ◯ True ◯ False

7. Let $(G, \cdot)$ be a group. $S \subset G$, such that $|S| = n$ and $\langle S \rangle = G$ (i.e., the elements in S generate G). Then G has only finitely many elements. ◯ True ◯ False

8. If G and H are groups such that $|G| = n$ and $|H| = m$, then $|G \times H| = n + m$.
◯ True ◯ False

9. $(\mathcal{F}(\mathbb{R}), \cdot)$ is a group with identity element $\varepsilon_1 : \mathbb{R} \to \mathbb{R}$ defined by $\varepsilon_1(x) = 1$.
◯ True ◯ False

10. $(\mathbb{Q}, +)$ is isomorphic to $(\mathbb{Z}, +)$. ◯ True ◯ False
Hint: Suppose $F : \mathbb{Q} \to \mathbb{Z}$ is an isomorphism. If $F(q) = 1$, what is $F(\frac{q}{2})$?

11. Let $p_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & 4 & 9 & 2 & 3 & 8 & 1 & 6 & 5 \end{pmatrix}$ be a permutation in $(S_9, \circ)$.
Then $p_1 = (17) \circ (24) \circ (68) \circ (395)$. ◯ True ◯ False

12. Let $p_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & 9 & 5 & 3 & 1 & 2 & 4 & 8 & 6 \end{pmatrix}$ be a permutation in $(S_9, \circ)$.
Then $p_2 = (43517) \circ (296)$. ◯ True ◯ False

13. Let $p_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 8 & 7 & 4 & 3 & 6 & 5 & 1 & 2 \end{pmatrix}$ be a permutation in $(S_9, \circ)$.
Then $p_3^{37} = p_3$. ◯ True ◯ False

14. For any two cycles $b, c \in (S_n, \circ)$ we have that $c \circ b = b \circ c$. ◯ True ◯ False

15. The set $S_{\mathbb{R}} = \{f : \mathbb{R} \to \mathbb{R}, f \text{ bijective } \}$ is a subgroup of $(\mathcal{F}(\mathbb{R}), +)$. ◯ True ◯ False

16. Let a be an element of order 12 in a group G. Then the order of a^8 is 4. ◯ True ◯ False

17. Let G be a group and let $a, b \in G$ with $a \in \langle b \rangle$. Then $\langle a \rangle = \langle b \rangle$ if and only if a and b have the same order. ◯ True ◯ False

Long answer questions

Question 1 *(5 points)* Let $(G, \cdot)$ be a group and $H = \langle\{a, b\}\rangle$ be the subgroup generated by the elements a and b, which satisfy the equations

$$a^2 = e \quad , \quad b^3 = e \quad , \quad ab = ba.$$

a) Show that H is an abelian group.

b) How many different elements can H contain at most?

Question 2 *(5 points)* Determine which of the following groups are isomorphic to which others. Prove your answers.

$$\mathbb{Z}_8, \quad P_3, \quad \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, \quad D_4$$

where P_3 is the group of subsets of a three element set.