Math 31: Exam 1 Practice

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Test your knowledge

True/false questions

1. $+_4$ is an operation on the set $\mathbb{Z}_2 = \{0, 1\}$. **False**

2. Let $*$ be an operation on a set A. If $(A, *)$ has a neutral element e, then e is unique. **True**

3. Let $\langle G, \cdot \rangle$ be a group and $a, b \in G$. If a and b commute, then a^2 commutes with b^2. **True**

4. Let $\langle G, \cdot \rangle$ be a group and H and K subgroups of G. Then $H \cup K$ is a subgroup of G. **False**

   Counterexample: Consider the group $\langle \mathbb{Z}, + \rangle$. This group has subgroups $\langle 2 \rangle$ and $\langle 3 \rangle$, but $\langle 2 \rangle \cup \langle 3 \rangle$ is not a subgroup since it contains both 2 and 3, but not $2 + 3 = 5$.

5. The set $H = \{f : \mathbb{R} \to \mathbb{R} \mid f(x) \geq 0 \text{ for all } x \in \mathbb{R}\}$ is a subgroup of $(\mathcal{F}(\mathbb{R}), +)$. **False**

6. Let $(G, \cdot)$ be a group, $a, b \in G$ fixed and $f : G \to G$ be the function defined by $f(x) = axb$. Then f is bijective. **True**

7. Let $(G, \cdot)$ be a group. $S \subset G$, such that $|S| = n$ and $\langle S \rangle = G$ (i.e., the elements in S generate G). Then G has only finitely many elements. **False**

   Counterexample: Note that $\{1\} \subset \mathbb{Z}$, $|\{1\}| = 1$, and $\langle \{1\} \rangle = \mathbb{Z}$.

8. If G and H are groups such that $|G| = n$ and $|H| = m$, then $|G \times H| = n + m$. **False**

9. $(\mathcal{F}(\mathbb{R}), \cdot)$ is a group with identity element $\varepsilon_1 : \mathbb{R} \to \mathbb{R}$ defined by $\varepsilon_1(x) = 1$. **False**

   Not every element has an inverse under multiplication. For example, let $f(x) = x$. If there is a function $f^{-1}(x)$ in this set, then $f \cdot f^{-1} = \varepsilon_1$. So we would have $\left[f \cdot f^{-1}\right](0) = \varepsilon_1(0)$, i.e., $0 \cdot f^{-1}(0) = 1$, which is a contradiction.

10. $(\mathbb{Q},+)$ is isomorphic to $(\mathbb{Z},+)$. **False**
Hint: Suppose $F:\mathbb{Q}\to\mathbb{Z}$ is an isomorphism. If $F(q)=1$, what is $F(\frac{q}{2})$?
Suppose $F:\mathbb{Q}\to\mathbb{Z}$ is an isomorphism. Then F is surjective and so there exists a rational number q such that $F(q)=1$. Since F is an isomorphism, we have that

$$F(\frac{q}{2})+F(\frac{q}{2})=F(\frac{q}{2}+\frac{q}{2})=F(q)=1.$$

In other words, $2F(\frac{q}{2})=1$ and so $F(\frac{q}{2})=1/2$. Since $1/2$ is not an integer, we have reached a contradiction.

11. Let $p_1=\begin{pmatrix}1&2&3&4&5&6&7&8&9\\7&4&9&2&3&8&1&6&5\end{pmatrix}$ be a permutation in $(S_9,\circ)$.
Then $p_1=(17)\circ(24)\circ(68)\circ(395)$. **True**

12. Let $p_2=\begin{pmatrix}1&2&3&4&5&6&7&8&9\\7&9&5&3&1&2&4&8&6\end{pmatrix}$ be a permutation in $(S_9,\circ)$.
Then $p_2=(43517)\circ(296)$. **True**

13. Let $p_3=\begin{pmatrix}1&2&3&4&5&6&7&8&9\\9&8&7&4&3&6&5&1&2\end{pmatrix}$ be a permutation in $(S_9,\circ)$.
Then $p_3^{37}=p_3$. **True**

14. For any two cycles $b,c\in(S_n,\circ)$ we have that $c\circ b=b\circ c$. **False**

15. The set $S_{\mathbb{R}}=\{f:\mathbb{R}\to\mathbb{R}, f \text{ bijective }\}$ is a subgroup of $(\mathcal{F}(\mathbb{R}),+)$. **False**
Red flag: $S_{\mathbb{R}}$ uses the operation of function composition, while $\mathcal{F}(\mathbb{R})$ uses the operation of function addition.
We can check that the sum of two bijective functions is not always bijective, and so $S_{\mathbb{R}}$ is not closed under the operation of $\mathcal{F}(\mathbb{R})$. For example, let $f(x)=x$ and $g(x)=-x$. Then $f,g\in S_{\mathbb{R}}$, but $[f+g](x)=x+(-x)=0$ for all x and so clearly is not injective or surjective.

16. Let a be an element of order 12 in a group G. Then the order of a^8 is 4. **False**
We know that $a^12=e$ and that all of the powers $a^0,a,a^2,\ldots,a^11$ are different. To find the order of a^8, repeatedly multiply it by itself until you get e:

$$(a^8)^1\neq e \text{ (since } \mathrm{ord}(a)=12)$$
$$(a^8)^2=a^{16}=a^{12+4}=a^{12}a^4=a^4\neq e$$
$$(a^8)^3=(a^8)^2a^8=a^4a^8=a^{12}=e.$$

17. Let G be a group and let $a,b\in G$ with $a\in\langle b\rangle$. Then $\langle a\rangle=\langle b\rangle$ if and only if a and b have the same order. **False**
This is false if G is an infinite group and a and b both have infinite order. This is true if the orders of a and b are both finite.

Long answer questions

Question 1 *(5 points)* Let $(G, \cdot)$ be a group and $H = \langle\{a, b\}\rangle$ be the subgroup generated by the elements a and b, which satisfy the equations

$$a^2 = e \quad , \quad b^3 = e \quad , \quad ab = ba.$$

a) Show that H is an abelian group.

Since H is generated by $\{a, b\}$, every element of H is a product of a, b, a^{-1}, and b^{-1}. Note that since $ab = ba$, we also have that

$$ab^{-1} = b^{-1}bab^{-1} = b^{-1}abb^{-1} = b^{-1}a,$$

$$ba^{-1} = a^{-1}aba^{-1} = a^{-1}baa^{-1} = a^{-1}b,$$

and

$$a^{-1}b^{-1} = (ba)^{-1} = (ab)^{-1} = b^{-1}a^{-1}.$$

Thus all of a, b, a^{-1}, and b^{-1} commute with each other. Therefore, since every element is a product of these elements, H is an abelian group.

b) How many different elements can H contain at most?

There are multiple ways to solve this one. Note that since $a^2 = e$, a is its own inverse. Since $b^3 = e$, b^2 is the inverse of b. So, $\{e, a, b, b^2\} \subset H$. Since $ab = ba$, each element in H can be written in the form $a^n b^m$. If n is even, then $a^n b^m = eb^m = b^m$ and so $a^n b^m$ is equal to either b, b^2, or e. If n is odd, then $a^n b^m = ab^m$ and so $a^n b^m$ is equal to either ab, ab^2, or a. So H contains at most the elements e, a, b, b^2, ab, ab^2. So $|H| \leq 6$.

Question 2 *(5 points)* Determine which of the following groups are isomorphic to which others. Prove your answers.

$$\mathbb{Z}_8, \quad P_3, \quad \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, \quad D_4$$

where P_3 is the group of subsets of a three element set.

All of these groups have order 8. Only $\mathbb{Z}_8$ is cyclic, so none of the others are isomorphic to it. The group D_4 has an element of order 4, while in P_3 and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ every element has order 2. (You showed in the first homework assignment that every set is its own inverse in P_D.) So D_4 is not isomorphic to any of the others. So we either need to show the remaining two are isomorphic, or find a difference in their group structure.

Claim: $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ is isomorphic to P_3.

Let $\{a, b, c\}$ be the three element set that P_3 is built from. Consider the function $f : \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \to P_3$ defined by $f(x, y, z)$ equals the set containing x a's, y b's, and z c's. For example, $f(0, 1, 1) = \{b, c\}$. This function is clearly a bijection, since x, y, z are all 0 or 1, and $P_3 = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$. We need to check that it satisfies the isomorphism property: $f((x, y, z) + (x', y', z')) = f(x, y, z) + f(x', y', z')$. We can see that this holds by considering the operations on each group. In $\mathbb{Z}_2$, if two elements are equal, then their sum is 0. Thus the corresponding element in $\{a, b, c\}$ does not appear in image of f. On the other hand, in P_3, if two sets contain the same element, then their sum does not contain that element (this is precisely the definition of addition here).