Homework 6

(Quotient rings, FHT for rings, prime ideals, maximal ideals, integral domains.)

Due Friday, November 8 at 2:10pm in class.

Note: Be sure to justify your answers. No credit will be given for answers without work/justification. In addition, all written homework assignments should be neat and well-organized; **Part A and Part B should be submitted separately**.

Part A:

(1) Chapter 19 A.2 and B.3 (You may use Chapter 18 E.6 from HW05.)

(2) Let A be a ring.

- An ideal J is called *semiprime* if, for all $a \in A$, if $a^n \in J$ for some positive integer n, then $a \in J$.
- An element $a \in A$ is called *nilpotent* if $a^n = \underbrace{a \cdot a \cdots a}_{n \text{ times}} = 0$ for some integer $n > 0$.

Prove each of the following statements. For each part, you may use the preceding parts even if you are unable to prove them.

(a) An ideal J of A is semiprime if and only A/J has no nilpotent elements (except 0).

(b) If A is an integral domain, then A has no nonzero nilpotent elements.

(c) Every prime ideal in a commutative ring is semiprime.

Part B:

(1) Prove each of the following statements. For each part, you may use the preceding parts even if you are unable to prove them.

(a) The only ideals in a field F are the trivial ideals $\{0\}$ and F.

(b) Let A and B be rings. Let $f : A \to B$ be a surjective homomorphism from A to B. If J is an ideal of A such that $\ker(f) \subset J$, then $f(J)$ is an ideal of B.

(c) Let A be a ring and B be a field. Let $f : A \to B$ be a surjective homomorphism from A to B. Then $\ker(f)$ is a maximal ideal.
(Hint: Use part (b), then part (a). In addition, consider the following: if $a \in A$ and $j \in J$ such that $f(a) = f(j)$, what can you say about $f(a - j)$?)

(d) If A/J is a field, then J is a maximal ideal.

(2) Chapter 19 D.1 and D.2.