Homework 7

(Finite fields, polynomial rings, factoring polynomials.)

Due Wednesday, November 13 at 2:10pm in class.

Note: Be sure to justify your answers. No credit will be given for answers without work/justification. In addition, all written homework assignments should be neat and well-organized; **Part A and Part B should be submitted separately**.

Part A:

(1) For this problem, you may use Theorem 3 in Chapter 20 without proof.
- (a) Suppose that A is a field, B is a ring, and $f : A \to B$ is a ring homomorphism. Show that if the range of f has more than one element, then f is an injective function.
- (b) If A is an integral domain with characteristic p, show that $f : A \to A$ defined by $f(a) = a^p$ is a homomorphism.
- (c) If A is a finite field of characteristic p, show that $f : A \to A$ defined by $f(a) = a^p$ is an isomorphism.

(2) Let $a(x) = 5x^2 + 6$ and $b(x) = 3x^3 + x^2 + 5$.
- (a) Compute $a(x)b(x)$ over $\mathbb{Z}_7$.
- (b) Compute $a(x)b(x)$ over $\mathbb{Z}_{15}$.
- (c) Use long division of polynomials to find the quotient $q(x)$ and remainder $r(x)$ when $b(x)$ is divided by $a(x)$.

Part B:

(1) Let A be a commutative ring with unity. Let J consist of all elements in $A[x]$ whose constant term is equal to 0. Show that J is a prime ideal of $A[x]$.

(2) Let A be a commutative ring with unity and $h : A[x] \to A$ be defined by

$$h(a_n x^n + a_{n-1}x^{n-1} + \cdots a_1 x + a_0) = a_0.$$

- (a) Show that h is a surjective homomorphism.
- (b) Show that $\ker(h)$ is the principal ideal $\langle x \rangle$.
- (c) Conclude that $A[x]/\langle x \rangle \cong A$.

(3) Let F be a field, and let J designate any ideal of $F[x]$. Show that there is a monic generator $m(x)$ of J such that

$$a(x) \in J \text{ if and only if } m(x) \mid a(x).$$