Handout: Proof skills

(1) Write the negation of each of the following statements.
   (a) The numbers a and b are in the set S.
   (b) Either a or b is in the set S.
   (c) There exists a group G which is not commutative.
   (d) Every integer is even.

(2) For each pair of statements p and q below, determine whether the statement "p and q" and the statement "p or q" are true or false.
   (a) $p =$ "Every group has an identity element."
   $q =$ "Every group is associative."
   (b) $p =$ "Every operation is associative."
   $q =$ "The set $\mathbb{Q}$ with the operation of subtraction forms a group."
   (c) $p =$ "The group $\mathbb{Z}_6$ has order 5."
   $q =$ "The group $\mathbb{Z}_2 \times \mathbb{Z}_4$ has 6 elements."

(3) Prove the following statement using a direct proof.

Theorem 1. *Let $a, x,$ and y be elements of a group G. If $xay = a^{-1}$, then $yax = a^{-1}$.*

(4) Prove the following statements using a proof by contradiction.

Theorem 2. *If $a, b \in \mathbb{Z}$, then $a^2 - 4b \neq 2$.*

Theorem 3. *The number $\sqrt{2}$ is irrational.*

Theorem 4. *There are infinitely many prime numbers.*

(5) Prove the following statement by proving its contrapositive.

Theorem 5. *Let x and y be integers. If $x + y$ is even, then x and y are both even or x and y are both odd.*

(6) Prove the following "if and only if" statement.

Theorem 6. *Suppose that a, b, and c are elements of a group G and $c = c^{-1}$. Then, $ab = c$ if and only if $abc = e$.*