Math 31: Topics in Algebra
Summer 2019 - Problem Set 1

Due: Wednesday, July 3

Instructions: Write or type your answers neatly, and staple all pages before passing in your assignment. Use complete sentences where appropriate. Show your work; little credit will be given for solutions without work or justification.

#1 Which of the following rules are binary operations on the indicated sets? For each rule that *is* a binary operation, determine whether or not the set is a group under this operation. Justify your answers.

a) $A = \mathbb{R} \setminus \mathbb{Q}$, $a * b = ab$ (ordinary multiplication).

b) $\mathrm{Mat}_2(\mathbb{R})^+ = \{\text{2-by-2 matrices } A \text{ with entries in } \mathbb{R} \mid \det(A) > 0\}$, $A * B = AB$ (matrix multiplication).

c) $\mathrm{Mat}_2(\mathbb{R})^{\mathrm{up}} = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\}$, $A * B = AB$ (matrix multiplication).

d) $A = \{-4, -3, -1, 1, 2, 3, 4, 5\}$, $a * b = |b|$.

#2 In class, we defined the *symmetric difference* as a binary operation defined on $\mathcal{P}(U)$ for any set U by

$$A \triangle B = (A \setminus B) \cup (B \setminus A).$$

Prove that this binary operation is associative.

#3 (Abelian groups)

a) Prove that if G is an abelian group, $a, b \in G$, then for all integers n, $(ab)^n = a^n b^n$.

b) Prove that if G is a group and for all $a, b \in G$, $(ab)^2 = a^2 b^2$, then G is abelian.

c) Prove that G is a group in which every element is its own inverse, G is abelian.

#4 (Order of group elements)

a) Let G be a group and $g \in G$ with $\mathrm{ord}(g) = km$ for positive integers k and m. Prove that $\mathrm{ord}(g^k) = m$.

---

b) Let G be a group and let $g \in G$. An element $b \in G$ is called a *conjugate* of a if there exists an element $x \in G$ such that $b = xax^{-1}$. Show that if b is a conjugate of a and $\text{ord}(a) = n$, then $\text{ord}(b) = n$.

#**5** (Subgroups)

a) Let H be a subgroup of G and $x \in G$. Define $xHx^{-1} = \{xhx^{-1} : h \in H\}$. Prove that xHx^{-1} is also a subgroup of G.

b) Let G be a group and define the *center* of G by $Z(G) = \{z \in G : gz = zg \text{ for all } g \in G\}$ (the subset of G that commutes with everything in G). Prove that $Z(G)$ is a subgroup of G.

c) What are all of the subgroups of $\mathbb{Z}_{12}$ under addition modulo 12?