Math 31: Topics in Algebra
Summer 2019 - Problem Set 2

Due: Wednesday, July 10

Instructions: Write or type your answers neatly, and staple all pages before passing in your assignment. Use complete sentences where appropriate. Show your work; little credit will be given for solutions without work or justification.

#1.

a) Write $\sigma = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 4 & 9 & 7 & 5 & 1 & 6 & 8 & 3 \end{smallmatrix}\right)$ in cycle notation.

b) Write $\tau = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & 3 & 8 & 5 & 1 & 4 & 6 & 9 & 2 \end{smallmatrix}\right)$ in cycle notation.

c) Compute $\sigma^{-1}\tau$. Write your answer in both cycle notation and two-line notation.

d) Write τ as a product of transpositions.

#2 Let $\sigma \in S_n$ be a permutation written in cycle notation as $C_1 \cdot C_2 \cdots C_k$, where each C_i is a cycle of length m_i and these cycles are disjoint. What is $\operatorname{ord}(\sigma)$? Prove your answer.

#3 (Generators for S_n)

a) Prove that any permutation in S_n can be written as a product of *adjacent transpositions*: transpositions of the form $(i \quad i+1)$ for $i \in \{1, 2, \ldots, n-1\}$. (Hint: we already know that σ can be written as a product of transpositions.)

b) Prove that if $\sigma \in S_n$ is a k-cycle $(\sigma_1 \ \sigma_2 \ \cdots \ \sigma_k)$ and $\pi \in S_n$, then $\pi\sigma\pi^{-1}$ is also a k-cycle. (Hint: first prove it's true if π is a cycle.)

c) Use a) and b) to prove that any permutation in S_n can be written as a product of the two permutations $(1\,2 \ \cdots \ n)$ and (12) (and their inverses).

#4 (Homomorphisms)

a) Let $f : G \to H$ be a group homomorphism. Show that as a function from G to H, f is injective if and only if $\operatorname{Ker}(f) = \{e_G\}$.

b) Let G be a group and denote by $\operatorname{Aut}(G)$ the set of isomorphisms from G to itself. Show that $\operatorname{Aut}(G)$ is a group under function composition.

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#**5**

a) Show that $\mathbb{Z} \cong E$ where E is the group of even integers under addition.

b) Let P_2 be the group of subsets of $\{a, b\}$ under the symmetric difference operation. Prove $P_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.